GROMOV-WITTEN INVARIANTS FOR G/B AND PONTRYAGIN PRODUCT FOR ΩK

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ABSTRACT. We give an explicit formula for (T-equivariant) 3-pointed genus zero Gromov-Witten invariants for G/B. We derive it by finding an explicit formula for the Pontryagin product on the equivariant homology of the based loop group ΩK.

1. INTRODUCTION

A flag variety G/B is the quotient of a simply-connected simple complex Lie group by its Borel subgroup and it plays very important roles in many different branches of mathematics. There are natural Schubert cycles inside G/B. The corresponding Schubert cocycles σ^u's form a basis of the cohomology ring $H^*(G/B)$. In terms of this basis, the structure coefficients $N_{u,v}^w$'s of the intersection product,

$$\sigma^u \cdot \sigma^v = \sum_w N_{u,v}^w \sigma^w,$$

are called *Schubert structure constants*, which is a direct generalization of the Littlewood-Richardson coefficients for complex Grassmannians. When $G = SL(n+1, \mathbb{C})$, these coefficients count suitable Young tableaus (see e.g. [10]) or honeycombs [19], [20]. An explicit formula for $N_{u,v}^w$ in all cases are given by Kostant and Kumar [21] by considering Kac-Moody groups and an effective algorithm is obtained by Duan [6] via topological methods. Note that a ring presentation of $H^*(G/B, \mathbb{C})$ is given much earlier by Borel [2] in terms of Chern classes of universal bundles over G/B.

The (small) quantum cohomology ring of G/B, or more generally of any symplectic manifold, is introduced by the physicist Vafa [38] and it is a deformation of the ring structure on $H^*(G/B)$ by incorporating genus zero Gromov-Witten invariants of G/B into the intersection product. As complex vector spaces, the quantum cohomology ring $QH^*(G/B)$ is isomorphic to $H^*(G/B) \otimes \mathbb{C}[\mathbf{q}]$ with $\mathbf{q}_\lambda = q_1^{a_1} \cdots q_n^{a_n}$ for $\lambda = (a_1, \cdots, a_n) \in H_2(G/B, \mathbb{Z})$. The structure coefficients $N_{u,v}^{w,\lambda}$'s of the quantum product,

$$\sigma^u \star \sigma^v = \sum_{w,\lambda} N_{u,v}^{w,\lambda} \mathbf{q}_\lambda \sigma^w,$$

are called *quantum Schubert structure constants*. As we will see in section 5.1, $N_{u,v}^{w,\lambda} = I_{0,3,\lambda}(\sigma^u, \sigma^v, \sigma^{\omega_0 w})$ is the 3-pointed genus zero Gromov-Witten invariant for $\sigma^u, \sigma^v, \sigma^{\omega_0 w} \in H^*(G/B)$, by the definition of the quantum product $\sigma^u \star \sigma^v$. We will use the terminology "quantum Schubert structure constants" instead of "Gromov-Witten invariants" for G/B throughout this paper, in analog with the classical Schubert structure constants.

Because of the lack of functoriality, the study of the quantum cohomology ring of G/B, or more generally partial flag varieties G/P, is a challenging problem. A presentation of the ring structure on $QH^*(G/B)$ is given by Kim [18] in terms of Toda lattice for the Langlands dual Lie group. There have been a lot of studies of $QH^*(G/P)$ in special cases including complex Grassmannians, partial flag varieties of type A, isotropic Grassmannians and two exceptional minuscule homogeneous varieties (see e.g. [3], [4], [22], [23] and [5] respectively and the excellent survey [9]). Nevertheless, the quantum Schubert structure constants had only been computed explicitly for very few cases, such as complex Grassmannians and complete flag varieties of type A.

In this article, we give an explicit formula for the (equivariant) quantum Schubert structure constants of the quantum cohomology ring $QH^*(G/B)$ (for partial flag varieties G/P, see [29]). We should note that an algorithm to determine the equivariant quantum Schubert structure constants[1] was obtained earlier by Mihalcea [33] and he used it to find a characterization of the $QH^*_T(G/P)$. To describe the formula, we define the rational functions $c_{x,[y]}$ and $d_{x,[y]}$ combinatorially for any $x, y \in W_{\rm af} = W \ltimes Q^\vee$. In particular for $x = ut_A, y = vt_A$ and $z = wt_{2A+\lambda}$ with $A = -12n(n+1)\sum_{i=1}^n w_i^\vee$ a sum of fundamental coweights $w_i^\vee$'s, the rational function $\sum_{\lambda_1,\lambda_2\in Q^\vee} c_{x,[t_{\lambda_1}]}c_{y,[t_{\lambda_2}]}d_{z,[t_{\lambda_1+\lambda_2}]}$ will be shown to be a constant, provided that $\langle\lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w)$. Furthermore, this number coincides with $N_{u,v}^{w,\lambda}$ as stated in our main theorem.

Main Theorem *Let $u, v, w \in W$, $\lambda \in Q^\vee, \lambda \succcurlyeq 0$. Let $A = -12n(n+1)\sum_{i=1}^n w_i^\vee$. The quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ for G/B is given by*

$$N_{u,v}^{w,\lambda} = \sum_{\lambda_1,\lambda_2\in Q^\vee} c_{ut_A,[t_{\lambda_1}]}c_{vt_A,[t_{\lambda_2}]}d_{wt_{2A+\lambda},[t_{\lambda_1+\lambda_2}]},$$

provided that $\langle\lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w)$ and zero otherwise.

The summation over the infinite set $Q^\vee$ can be simplified to the finite set $\Gamma\times W$ with $\Gamma = \{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \succcurlyeq A, \lambda_1 + \lambda_2 \preccurlyeq 2A + \lambda, \lambda_1, \lambda_2 \in \tilde{Q}^\vee\}$ and we obtain

$$N_{u,v}^{w,\lambda} = \sum_{(\lambda_1,\lambda_2,v_1)\in\Gamma\times W} c_{ut_A,[v_1t_{\lambda_1}]}c_{vt_A,[v_1t_{\lambda_2}]}d_{wt_{2A+\lambda},[v_1t_{\lambda_1+\lambda_2}]}.$$

The choice of A is not unique. In many cases, we can replace it by a *smaller* one (see the proof of Theorem 5.1). All these and their T-equivariant extensions will be given in section 5.2.

As a consequence, the above summations have only a few nonzero terms in many cases. For instance for $G = SL(3,\mathbb{C})$ with $u = v = s_1s_2s_1, w = s_1s_2$ and $\lambda = \theta^\vee$ (see section 6.1 for notations), it suffices to take $A = -\theta^\vee$ and the summation for $N_{u,v}^{w,\lambda}$ in fact contains one term only, namely $N_{u,v}^{w,\lambda} = c^2_{s_0,[s_0]}d_{s_2s_0,[s_1s_2s_1t_{-2\theta^\vee}]}$, where $c_{s_0,[s_0]} = (-1)^1\frac{1}{s_0(\alpha_0)}\big|_{\alpha_0=-\theta} = -\frac{1}{\theta}$ and $d_{s_2s_0,[s_1s_2s_1t_{-2\theta^\vee}]} = d_{s_2s_0,[s_0s_1s_2s_1s_0]} = s_0s_1(\alpha_2)s_0s_1s_2s_1(\alpha_0)\big|_{\alpha_0=-\theta} = \theta^2$ by definition. Hence, $N_{u,v}^{w,\lambda} = (-\frac{1}{\theta})^2\cdot\theta^2 = 1$. This coefficient can also be determined by Mihalcea's algorithm but our formula is more effective. To show the computational power of our formula, we will compute some nontrivial coefficients for the higher rank group $Spin(7,\mathbb{C})$.

[1] Explicitly, the equivariant (quantum) Schubert structure constants are homogeneous polynomials.

Quantum Schubert structure constants for G/P can be identified with certain quantum Schubert structure constants for G/B via Peterson-Woodward comparison formula [39], the corresponding formula and its applications will be discussed in [29].

When v is a simple reflection, the equivariant quantum product $\sigma^u \star \sigma^v$ can be given explicitly by the equivariant quantum Chevalley formula. This formula was originally stated by Peterson in his unpublished work [35] and has been proved recently by Mihalcea [33]. In [33], Mihalcea also showed that the multiplication in $QH_T^*(G/B)$ is determined by the equivariant quantum Chevalley formula together with a few other natural properties (see e.g. Proposition 5.3). As a consequence, an algorithm to determine $N_{u,v}^{w,\lambda}$ was given in [33]. However, an explicit formula is still lacking.

In [35] Peterson already realized that $QH_T^*(G/B)$ should be ring isomorphic to $H_*^T(\Omega K)$ after localization. Here ΩK is the based loop group of the maximal compact subgroup K of G and the *Pontryagin product* defines a ring structure on its (Borel-Moore) homology group $H_*^T(\Omega K)$. In Peterson's work [35], the powerful tool of nil-Hecke ring of Kostant-Kumar [21] was used heavily. Lam and his co-authors had done many important works along this direction, such as [25], [28], [26] and [27]. In [26], Lam and Shimozono gave a proof of the canonical isomorphism between $QH_T^*(G/B)$ and $H_*^T(\Omega K)$ after localization, with the help of Peterson's j-isomorphism. Our approach is similar to the one by Lam-Shimozono as we both rely heavily on Mihalcea's criterion [33].

The homology $H_*^T(\Omega K)$ is an associative algebra over $S = H_T^*(\mathrm{pt})$ and it has an additive S-basis given by Schubert homology classes $\{\mathfrak{S}_x \mid x \in W_{\mathrm{af}}^-\}$, where W_{af}^- is the set of length-minimizing representatives of cosets in W_{af}/W. We obtain an explicit formula for the Pontryagin product of Schubert classes in $H_*^T(\Omega K)$.

Theorem 3.3 *For any Schubert classes $\mathfrak{S}_x$ and $\mathfrak{S}_y$ in $H_*^T(\Omega K)$, the structure coefficients for their Pontryagin product*

$$\mathfrak{S}_x\mathfrak{S}_y = \sum_{z\in W_{\mathrm{af}}^-} b_{x,y}^z \mathfrak{S}_z$$

are given by

$$b_{x,y}^z = \sum_{\lambda,\mu\in Q^\vee} c_{x,[t_\lambda]} c_{y,[t_\mu]} d_{z,[t_{\lambda+\mu}]}.$$

By analyzing the combinatorial nature of the summation in our formula, we obtain elementary proofs of the following two formulas of Peterson-Lam-Shimozono [26] on the Pontryagin product of certain Schubert classes: (i) *For any $wt_\lambda, t_\mu \in W_{\mathrm{af}}^-$, one has $\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu} = \mathfrak{S}_{wt_{\lambda+\mu}}$*; (ii) *For any $\sigma_i t_\lambda, ut_\mu \in W_{\mathrm{af}}^-$ with $\sigma_i = \sigma_{\alpha_i}$, $i \in I$, one has*

$$\mathfrak{S}_{\sigma_i t_\lambda}\mathfrak{S}_{ut_\mu} = (u(w_i) - w_i)\mathfrak{S}_{ut_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_1} \langle\gamma^\vee, w_i\rangle \mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_2} \langle\gamma^\vee, w_i\rangle \mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}},$$

where $\Gamma_1 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1\}$ and $\Gamma_2 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1 - \langle\gamma^\vee, 2\rho\rangle\}$.

By combining these with the criterion of Mihalcea, our above formulas for the structure coefficients of the Pontryagin product on $H_*^T(\Omega K)$ imply the corresponding formula for the equivariant quantum Schubert structure constants for $QH_T^*(G/B)$.

There could be an alternative way to determine our structure coefficients by finding polynomial representatives for Schubert classes. For instance, this approach has been used by Fomin, Gelfand and Postnikov for complete flag varieties of type A [8]. The work of Magyar [31] could be relevant for general cases. See also [7].

This paper is organized as follows. In section 2, we set up the notations that will be used throughout this article and review some well-known facts on the theory of Kac-Moody algebras and groups. In section 3, we define the important quantities $c_{x,[y]}$, $d_{x,[y]}$ and derive an explicit formula for the Pontryagin product on $H_*^T(\Omega K)$. In section 4, we analyze our formula and compute the product of certain Schubert classes. In section 5, we prove our main theorem and we also show that the classical limit of our formula recover the usual formula for the equivariant Schubert structure constants . In section 6, we give examples to demonstrate the effectiveness of our formula. Finally in the appendix, we provide the proofs of some propositions stated in section 4.

2. Notations

2.1. **Notations.** We introduce the notations that are used throughout the paper.

G: a simply-connected simple complex Lie group of rank n.
B, H: B is a Borel subgroup of G; H is a maximal torus of G contained in B.
K: a maximal compact subgroup of G.
T: $T = K \cap H$ is a maximal torus in K.
$\mathfrak{g}, \mathfrak{h}$: $\mathfrak{g} = \mathrm{Lie}(G)$; $\mathfrak{h} = \mathrm{Lie}(H)$.
I, I_{af}: $I = \{1, \cdots, n\}$; $I_{\mathrm{af}} = \{0, 1, \cdots, n\}$.
R, Δ: R is the root system of $(\mathfrak{g}, \mathfrak{h})$; $\Delta = \{\alpha_i \mid i \in I\}$ is a basis of simple roots.
R^+: $R^+ = R \cap \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the set of the positive roots; $R = \big(-R^+\big) \bigsqcup R^+$.
$\alpha_i^\vee, Q^\vee$: $\{\alpha_i^\vee \mid i \in I\}$ are the simple coroots; $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ is the coroot lattice.
$\tilde{Q}^\vee$: $\tilde{Q}^\vee = \{\mu \in Q^\vee \mid \langle \mu, \alpha_i\rangle \leq 0,\ i \in I\}$ is the set of anti-dominant elements.
w_i, ρ: $\{w_i \mid i \in I\}$ are the fundamental weights; $\rho = \frac{1}{2}\sum_{\beta\in R^+} \beta\ \big(= \sum_{i\in I} w_i\big)$.
$w_i^\vee$: $\{w_i^\vee \mid i \in I\}$ are the fundamental coweights.
W: $W = \langle \sigma_{\alpha_i}\ :\ i \in I\rangle$ is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.
θ, ω_0: θ is the highest (long) root of R; ω_0 is the longest element in W.
$\mathfrak{g}_{\mathrm{af}}$: the (untwisted) affine Kac-Moody algebra associated to $\mathfrak{g}$.
$\mathfrak{h}_{\mathrm{af}}$: Cartan subalgebra of $\mathfrak{g}_{\mathrm{af}}$.
α_0, δ: α_0 is the affine simple root; $\delta = \alpha_0 + \theta$ is the null root.
R^+_{re}: $R^+_{\mathrm{re}} = \{\alpha + m\delta \mid \alpha \in R, m \in \mathbb{Z}^+\} \cup R^+$ is the set of positive real roots.
$\mathcal{S}, Y$: $\mathcal{S} = \{\sigma_{\alpha_i} \mid i \in I_{\mathrm{af}}\}$; $Y \subset \Delta$ is a subset.
W_{af}: the Weyl group of $\mathfrak{g}_{\mathrm{af}}$; $W_{\mathrm{af}} = \langle \sigma_{\alpha_i} : i \in I_{\mathrm{af}}\rangle$.
$W_{\mathrm{af},Y}$: the subgroup of W_{af} generated by $\{\sigma_\alpha \mid \alpha \in Y\}$.
W^Y_{af}: the subset $\{x \in W_{\mathrm{af}} \mid \ell(x) \leq \ell(y), \forall y \in xW_{\mathrm{af},Y}\}$ of W_{af}.
$\mathcal{G}$: the Kac-Moody group associated to the Kac-Moody algebra $\mathfrak{g}_{\mathrm{af}}$.
$\mathcal{B}$: the standard Borel subgroup of $\mathcal{G}$.
$\mathcal{P}_Y$: the standard parabolic subgroup of $\mathcal{G}$ associated to Y; $\mathcal{P}_Y \supset \mathcal{B}$.
$W^-_{\mathrm{af}}, \mathcal{P}_0$: $W^-_{\mathrm{af}} = W^\Delta_{\mathrm{af}}$; $\mathcal{P}_0 = \mathcal{P}_\Delta$.
$LK, \Omega K$: $LK = \{f : \mathbb{S}^1 \to K \mid f \text{ is smooth }\}$; $\Omega K = \{f \in LK \mid f(1_{\mathbb{S}^1}) = 1_K\}$.
$S, \hat{S}$: $S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$; $\hat{S} = \mathbb{Q}[\alpha_0, \alpha_1, \cdots, \alpha_n]$.
q_λ: $q_\lambda = q_1^{a_1} \cdots q_n^{a_n}$ for $\lambda = \sum_{i=1}^n a_i\alpha_i^\vee \in Q^\vee$.
σ_i, σ_β: $\sigma_i = \sigma_{\alpha_i}$ is a simple reflection. σ_β is a reflection for $\beta \in R^+_{\mathrm{re}} \bigsqcup \big(-R^+_{\mathrm{re}}\big)$.
σ_u, σ^u: Schubert classes for G/B, where $u \in W$.

$\mathfrak{S}_x, \mathfrak{S}^x$: Schubert classes for $\mathcal{G}/\mathcal{P}_Y$.
$c_{x,[y]}$: defined in section 3.1 for any $x, y \in W_{\mathrm{af}}^-$.
$d_{x,[y]}$: defined in section 3.1 for any $x, y \in W_{\mathrm{af}}^-$.
$c'_{x,y}$: $c'_{x,y} = c_{x,y}\big|_{\alpha_0=-\theta}$ with $c_{x,y}$ defined in section 3.1.
Γ_1: $\Gamma_1(u) = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u)+1\}$, or simply $\Gamma_1 = \Gamma_1(u)$.
Γ_2: $\Gamma_2(u) = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u)+1-\langle\gamma^\vee, 2\rho\rangle\}$, or simply $\Gamma_2 = \Gamma_2(u)$.

2.2. **Some more explanations.** See [17] and [24] for the meaning of the notations as in section 2.1 as well as the theory of Kac-Moody algebras and groups.

The fundamental weights $\{w_i \mid i \in I\}$ are the dual basis to the simple roots $\{\alpha_i^\vee \mid i \in I\}$ with respect to the natural pairing $\langle\cdot,\cdot\rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$. The simple reflections $\{\sigma_i = \sigma_{\alpha_i} \mid i \in I\}$ act on $\mathfrak{h}$ by $\sigma_i(\lambda) = \lambda - \langle\lambda, \alpha_i\rangle\alpha_i^\vee$ for $\lambda \in \mathfrak{h}$. Therefore the Weyl group W, which is generated by the simple reflections, acts on $\mathfrak{h}$ and $\mathfrak{h}^*$ naturally. Note that $R = W \cdot \Delta$. For any $\gamma \in R$, $\gamma = w(\alpha_i)$ for some $w \in W$ and $i \in I$. We can well define $\gamma^\vee = w(\alpha_i^\vee)$, which is independent of the expressions of γ.

The Weyl group W_{af} of $\mathfrak{g}_{\mathrm{af}}$ is in fact an affine group, $W_{\mathrm{af}} = W \ltimes Q^\vee$, where we denote t_λ[2] the image of $\lambda \in Q^\vee$ in W_{af} (by abusing notations). To be more precise, one has $\sigma_\beta = \sigma_\alpha t_{m\alpha^\vee}$ for $\beta = \alpha + m\delta \in R_{\mathrm{re}} = \big(-R_{\mathrm{re}}^+\big) \bigsqcup R_{\mathrm{re}}^+$. In particular, $\sigma_{\alpha_0} = \sigma_\theta t_{-\theta^\vee}$. Given $w \in W, \lambda \in Q^\vee, \gamma \in \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ and $m \in \mathbb{Z}$, we have $t_{w\cdot\lambda} = wt_\lambda w^{-1}$ and the following action

$$wt_\lambda \cdot (\gamma + m\delta) = w \cdot \gamma + (m - \langle\lambda, \gamma\rangle)\delta.$$

Since $\big(W_{\mathrm{af}}, \mathcal{S}\big)$ is a Coxeter system, we can define the length function $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ and the Bruhat order $(W_{\mathrm{af}}, \preccurlyeq)$ (see e.g. [16]). We use the following notation

$$x = [\sigma_{\beta_1} \cdots \sigma_{\beta_r}]_{\mathrm{red}},$$

whenever $(\sigma_{\beta_1}, \cdots, \sigma_{\beta_r})$ is a reduced decomposition of $x \in W_{\mathrm{af}}$; that is, $r = \ell(x)$, $x = \sigma_{\beta_1} \cdots \sigma_{\beta_r}$ and β_i's are simple roots. (It is possible that $\beta_i = \beta_j$ for $i \neq j$.) This notation will also be used throughout this article.

Explicitly, the affine Kac-Moody group $\mathcal{G}$ is realized as a central extension by $\mathbb{C}^*$ of the loop group consisting of the $\mathbb{C}((t))$-rational points $G(\mathbb{C}((t)))$ of G extended by one dimensional complex torus. For each subset $Y \subset \Delta$, there is a standard parabolic subgroup $P_Y \subset \mathcal{G}$ corresponding to Y. In particular, $\mathcal{B} = \mathcal{P}_\emptyset$ and we denote $\mathcal{P}_0 = \mathcal{P}_\Delta$. For our purpose of studying the generalized flag varieties $\mathcal{G}/\mathcal{B}$ and $\mathcal{G}/\mathcal{P}_0$, the group $\mathcal{G}$ can be taken simply to be $\mathcal{G} = G(\mathbb{C}((t)))$. That is, $\mathcal{G} = \mathrm{Mor}(\mathbb{C}^*, G)$. As a consequence, $\mathcal{P}_0 = G(\mathbb{C}[[t]]) = \mathrm{Mor}(\mathbb{C}, G)$ and $\mathcal{B} = \{f \in \mathcal{P}_0 \mid f(0) \in B\}$.

In the present paper, we only consider the following two cases: $Y = \emptyset$ and $Y = \Delta$. Note that $W_{\mathrm{af},\emptyset} = \{1\}$, $W_{\mathrm{af}}^\emptyset = W_{\mathrm{af}}$ and $W_{\mathrm{af},\Delta} = W$. We denote $W_{\mathrm{af}}^- = W_{\mathrm{af}}^\Delta$.

3. Pontryagin product on equivariant homology of ΩK

The T-equivariant (Borel-Moore) homology $H_*^T(\Omega K)$ of based loop group ΩK is a module over $S = H_T^*(\mathrm{pt}) = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ with an S-basis of Schubert classes $\{\mathfrak{S}_x \mid x \in W_{\mathrm{af}}^-\}$, where W_{af}^- is the set of length-minimizing representatives of cosets in W_{af}/W. T acts on ΩK by pointwise conjugation. The Pontryagin product $\Omega K \times \Omega K \to \Omega K$, given by $(f \cdot g)(t) = f(t) \cdot g(t)$, is associative and T-equivariant.

[2] The notation t_λ is used instead of $t_{\nu(\lambda)}$ as in chapter 6 of [17].

Therefore, it induces a product map $H_*^T(\Omega K)\otimes H_*^T(\Omega K)\to H_*^T(\Omega K)$, making $H_*^T(\Omega K)$ an associative S-algebra. The structure constants $b_{x,y}^z\in S$ are defined by

$$\mathfrak{S}_x\mathfrak{S}_y=\sum\nolimits_{z\in W_{\rm af}^-}b_{x,y}^z\mathfrak{S}_z$$

for $x,y\in W_{\rm af}^-$. The main result of this section is Theorem 3.3, which gives an explicit formula for the Pontryagin product as $b_{x,y}^z=\sum_{\lambda,\mu\in Q^\vee}c_{x,[t_\lambda]}c_{y,[t_\mu]}d_{z,[t_{\lambda+\mu}]}$ with $c_{x,[y]},d_{x,[y]}$ defined combinatorially as below. As we will show in section 5.2, these structure coefficients $b_{x,y}^z$'s indeed correspond to quantum Schubert structure constants for the equivariant quantum cohomology $QH_T^*(G/B)$.

3.1. **Definitions and properties of $c_{x,[y]}$ and $d_{x,[y]}$.**

Definition 3.1. *For any $x,y\in W_{\rm af}$, we define the homogeneous rational function $c_{x,y}=c_{x,y}(\alpha_0,\cdots,\alpha_n)\in\mathbb{Q}[\alpha_0^{\pm},\alpha_1^{\pm},\cdots,\alpha_n^{\pm}]$ as follows. Let $x=[\sigma_{\beta_1}\cdots\sigma_{\beta_m}]_{\rm red}$. If $y\not\preccurlyeq x$, then $c_{x,y}=0$; if $y\preccurlyeq x$, then*

$$c_{x,y}\triangleq(-1)^m\sum\Big(\sigma_{\beta_1}^{\varepsilon_1}(\beta_1)\sigma_{\beta_1}^{\varepsilon_1}\sigma_{\beta_2}^{\varepsilon_2}(\beta_2)\cdots\sigma_{\beta_1}^{\varepsilon_1}\cdots\sigma_{\beta_m}^{\varepsilon_m}(\beta_m)\Big)^{-1},$$

where the summation runs over all $(\varepsilon_1,\cdots,\varepsilon_m)\in\{0,1\}^m$ satisfying $\sigma_{\beta_1}^{\varepsilon_1}\cdots\sigma_{\beta_m}^{\varepsilon_m}=y$.

We define $c_{x,[y]}\in\mathbb{Q}[\alpha_1^{\pm},\cdots,\alpha_n^{\pm}]$ as follows.

$$c_{x,[y]}\triangleq\sum\nolimits_{z\in yW}c_{x,z}\big|_{\alpha_0=-\theta}=\sum\nolimits_{z\in yW}c_{x,z}(-\theta,\alpha_1,\cdots,\alpha_n).$$

Let γ_k denote the (positive real) root $\sigma_{\beta_1}\cdots\sigma_{\beta_{k-1}}(\beta_k)$. We define the homogeneous polynomial $d_{y,x}=d_{y,x}(\alpha_0,\alpha_1,\cdots,\alpha_n)\in\mathbb{Q}[\alpha_0,\cdots,\alpha_n]$ as follows. If $y\not\preccurlyeq x$ then $d_{y,x}=0$; if $y=1$, then $d_{y,x}=1$; if $y\preccurlyeq x$ and $y\neq 1$, then

$$d_{y,x}\triangleq\sum\gamma_{i_1}\cdots\gamma_{i_r},$$

where the summation runs over all subsequences $(i_1,\cdots,i_r)$ of $(1,\cdots,m)$ such that $y=[\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_r}}]_{\rm red}$.

We define $d_{y,[x]}\in\mathbb{Q}[\alpha_1,\cdots,\alpha_n]$ as follow.

$$d_{y,[x]}\triangleq d_{y,x}\big|_{\alpha_0=-\theta}=d_{y,x}(-\theta,\alpha_1,\cdots,\alpha_n).$$

Note that for any $y,y'\in W_{\rm af}$ with $yW=y'W$, one has $c_{x,[y]}=c_{x,[y']}$. In addition, one has $d_{x,[y]}=d_{x,[y']}$ provided $x\in W_{\rm af}^-$ (following from Lemma 4.22).

Proposition 3.2 ([21]; see also chapter 11 of [24])**.** *$c_{x,y}$ and $d_{y,x}$ are well-defined, independent of the choices of reduced decompositions of x. The transpose of $\big(c_{x,y}\big)$ is the inverse of the matrix $\big(d_{x,y}\big)$ in the following sense*

$$\sum\nolimits_{z\in W_{\rm af}}c_{x,z}d_{y,z}=\delta_{x,y}=\sum\nolimits_{z\in W_{\rm af}}c_{z,x}d_{z,y},\quad\text{for any } x,y\in W_{\rm af}.$$

Note that both summations in the above proposition contain only finitely many nonzero terms.

3.2. **Explicit formula for Pontryagin product on $H_*^T(\Omega K)$.**

Because of the homotopy-equivalence between $\mathcal{G}/\mathcal{P}_0$ and ΩK, we interchange the notations $\mathcal{G}/\mathcal{P}_0$ and ΩK freely. Let $\hat T_{\mathbb{C}}$ denote the standard maximal torus of $\mathcal{G}$ with maximal compact sub-torus $\hat T$. The $\hat T$-equivariant cohomology $H_{\hat T}^*(\mathcal{G}/\mathcal{P}_0)$ is an $\hat S$-algebra with a basis of Schubert classes $\{\hat{\mathfrak{S}}^x\mid x\in W_{\rm af}^-\}$, where $\hat S=H_{\hat T}^*(\mathrm{pt})=\mathbb{Q}[\alpha_0,\alpha_1,\cdots,\alpha_n]$. Note that $T\subset\hat T$ is a sub-torus. The T-equivariant cohomology

$H_T^*(\mathcal{G}/\mathcal{P}_0)$ is an S-algebra with a basis of Schubert classes $\{\mathfrak{S}^x \mid x \in W_{\rm af}^-\}$, where $S = H_T^*(\mathrm{pt}) = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$. Furthermore, one has the following evaluation maps $\mathrm{ev} : H_{\hat{T}}^*(\mathcal{G}/\mathcal{P}_0) \to H_T^*(\mathcal{G}/\mathcal{P}_0)$ and $ev : \hat{S} \to S$ such that $\mathrm{ev}(f\hat{\mathfrak{S}}^x) = ev(f)\mathfrak{S}^x$, where $f = f(\alpha_0, \alpha_1, \cdots, \alpha_n) \in \hat{S}$ and $ev(f) = f(-\theta, \alpha_1, \cdots, \alpha_n) \in S$. (See appendix 7.4 for more details on the above descriptions.)

The T-equivariant homology $H_*^T(\mathcal{G}/\mathcal{P}_0)$ is the submodule of $\mathrm{Hom}_S(H_T^*(\mathcal{G}/\mathcal{P}_0), S)$ spanned by the equivariant Schubert homology classes $\{\mathfrak{S}_x \mid x \in W_{\rm af}^-\}$[3], which for any $x, y \in W_{\rm af}^-$ satisfy $\langle \mathfrak{S}_x, \mathfrak{S}^y\rangle = \delta_{x,y}$ with respect to the natural pairing.

The adjoint action of T on K induces a canonical action on ΩK by pointwise conjugation; that is, $(t \cdot f)(s) \triangleq t \cdot f(s) \cdot t^{-1}$ for any $t \in T$ and $f \in \Omega K$. The group multiplication $K \times K \to K$ induces a so-called Pontryagin product $\Omega K \times \Omega K \to \Omega K$ by pointwise multiplication; that is, $(f \cdot g)(s) = f(s) \cdot g(s)$ for any $f, g \in \Omega K$. The Pontryagin product is obviously associative and T-equivariant. Therefore it induces $H_*^T(\Omega K) \otimes H_*^T(\Omega K) \to H_*^T(\Omega K)$, which is also called the Pontryagin product. As a consequence, $H_*^T(\Omega K)$ is an associative S-algebra. Furthermore, one has

$$\mathfrak{S}_x\mathfrak{S}_y = \sum\nolimits_{z \in W_{\rm af}^-} b_{x,y}^z \mathfrak{S}_z$$

for $x, y \in W_{\rm af}^-$ with the structure coefficients $b_{x,y}^z \in S$ being polynomials. Now we state the main result of this section as follows.

Theorem 3.3. *For any $x, y, z \in W_{\rm af}^-$, the structure coefficient $b_{x,y}^z$ for $H_*^T(\Omega K)$ is given by*

$$b_{x,y}^z = \sum_{\lambda,\mu \in Q^\vee} c_{x,[t_\lambda]} c_{y,[t_\mu]} d_{z,[t_{\lambda+\mu}]}.$$

In particular, $b_{x,y}^z = b_{y,x}^z$ for all z, which implies $\mathfrak{S}_x\mathfrak{S}_y = \mathfrak{S}_y\mathfrak{S}_x$.

Note that $\mathcal{P}_0 = \mathcal{P}_\Delta$ and $W_{\rm af}^- = W_{\rm af}^\Delta$. By replacing Δ with a general subset $Y \subset \Delta$, $H_*^T(\mathcal{G}/\mathcal{P}_Y)$ and $\mathfrak{S}_Y^x$ can be defined in a similar manner. To distinguish these with the case of our main interest $\mathcal{G}/\mathcal{P}_0$, we denote $\mathfrak{S}_0^x, \mathfrak{S}_x^0$ for the case $Y = \emptyset$ (note that $\mathcal{P}_\emptyset = \mathcal{B}$). These notions can be extended to $\hat{T}$-equivariant (co)homology for $\mathcal{G}/\mathcal{P}_Y$ for a larger $\hat{T}$-action. The corresponding Schubert classes are denoted by "$\hat{\mathfrak{S}}$" instead of "$\mathfrak{S}$".

Definition 3.4. *Given $x \in W_{\rm af}$, we define the element ψ_x^Y in $\mathrm{Hom}_S(H_T^*(\mathcal{G}/\mathcal{P}_Y), S)$ to be the canonical morphism $\psi_x^Y \triangleq (\iota_x^Y)^* : H_T^*(\mathcal{G}/\mathcal{P}_Y) \to H_T^*(\mathrm{pt}) = S$, where ι_x^Y is the T-equivariant map $\iota_x^Y : \mathrm{pt} \to \mathcal{G}/\mathcal{P}_Y$ given by $\mathrm{pt} \mapsto x\mathcal{P}_Y$.*

We define $\hat{\psi}_x^Y$ to be the element $\hat{\psi}_x^Y = (\iota_x^Y)^*$ in $\mathrm{Hom}_{\hat{S}}(H_{\hat{T}}^*(\mathcal{G}/\mathcal{P}_Y), \hat{S})$, by considering the action by the larger group $\hat{T}$.

Since we consider the cases $Y = \Delta$ and $Y = \emptyset$ only, we simply denote

$$\psi_x = \psi_x^\Delta; \quad \hat{\psi}_x = \hat{\psi}_x^\Delta; \quad \psi_x^0 = \psi_x^\emptyset; \quad \hat{\psi}_x^0 = \hat{\psi}_x^\emptyset.$$

The relation between $\hat{\mathfrak{S}}_x^0$ and $\hat{\psi}_y^0$ is expressed in the following lemma.

Lemma 3.5 (Proposition 3.3.1 of [1][4])**.** *For any* $x \in W_{\rm af}$, $\hat{\mathfrak{S}}_x^0 = \sum_{y \in W_{\rm af}} c_{x,y}\hat{\psi}_y^0$.

[3]We should note that we are using the equivariant Borel-Moore homology (see e.g. [13]).

[4]The terminologies used in [1] and the present paper can be identified as follows: $\mathcal{L}_x = \hat{\mathfrak{S}}_x^0$ and $\Theta(\mu)(y) = \hat{\psi}_y^0(\mu)$ for $\mu \in H_{\hat{T}}^*(\mathcal{G}/\mathcal{B})$.

Note that one has $\psi_x = \psi_y$ whenever $xW = yW$ (following the definition), and that the canonical map $\pi_* : H_*^T(\mathcal{G}/\mathcal{B}) \to H_*^T(\mathcal{G}/\mathcal{P}_0)$, induced by the natural projection $\pi : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_0$, is given by

$$\pi_*(\mathfrak{S}_x^0) = \begin{cases} 0, & \text{if } x \notin W_{\text{af}}^- \\ \mathfrak{S}_x, & \text{if } x \in W_{\text{af}}^- \end{cases}.$$

Proposition 3.6. (i) *For any* $x \in W_{\text{af}}$, $\psi_x^0 = \sum_{y\in W_{\text{af}}} d_{y,[x]}\mathfrak{S}_y^0$ *in* $H_*^T(\mathcal{G}/\mathcal{B})$.

(ii) *For any* $x \in W_{\text{af}}^-$, $\psi_x = \sum_{y\in W_{\text{af}}^-} d_{y,[x]}\mathfrak{S}_y$ *in* $H_*^T(\mathcal{G}/\mathcal{P}_0)$.

(iii) *For any* $x \in W_{\text{af}}^-$, $\mathfrak{S}_x = \sum_{t\in Q^\vee} c_{x,[t]}\psi_t$.

Proof. (i) It follows from Proposition 3.2 and Lemma 3.5 that

$$\hat{\psi}_x^0 = \sum_{z\in W_{\text{af}}} \delta_{x,z}\hat{\psi}_z^0 = \sum_{z\in W_{\text{af}}}\sum_{y\in W_{\text{af}}} d_{y,x}c_{y,z}\hat{\psi}_z^0 = \sum_{y\in W_{\text{af}}} d_{y,x}\hat{\mathfrak{S}}_y^0.$$

Note that $\iota_x^\emptyset$ is both T-equivariant and $\hat{T}$-equivariant and id_{pt}, $\text{id}_{\mathcal{G}/\mathcal{B}}$ are morphisms preserving the T-action. Hence, the identity $\text{id}_{\mathcal{G}/\mathcal{B}}\circ\iota_x^\emptyset = \iota_x^\emptyset\circ\text{id}_{\text{pt}}$ induces $\psi_x^0 \circ \text{ev} = ev \circ \hat{\psi}_x^0 : H_{\hat{T}}^*(\mathcal{G}/\mathcal{B}) \to S$. Therefore,

$$\psi_x^0(\mathfrak{S}_0^y) = \psi_x^0 \circ \text{ev}(\hat{\mathfrak{S}}_0^y) = ev \circ \hat{\psi}_x^0(\hat{\mathfrak{S}}_0^y) = ev(d_{y,x}) = d_{y,[x]}.$$

Hence, $\psi_x^0 = \sum_{y\in W_{\text{af}}} d_{y,[x]}\mathfrak{S}_y^0$, where the summation contains only finitely many nonzero terms since $d_{y,[x]} = 0$ whenever $y \not\leq x$. Thus $\psi_x^0 \in H_*^T(\mathcal{G}/\mathcal{B})$.

(ii) Note that ι_x^Δ, $\iota_x^\emptyset$ and π are all T-equivariant maps. Hence, the identity $\iota_x^\Delta = \pi\circ\iota_x^\emptyset$ induces $\psi_x = \psi_x^0 \circ \pi^* : H_T^*(\mathcal{G}/\mathcal{P}_0) \to S$. Therefore,

$$\psi_x = \pi_*(\psi_x^0) = \pi_*(\sum\nolimits_{y\in W_{\text{af}}} d_{y,[x]}\mathfrak{S}_y^0) = \sum\nolimits_{y\in W_{\text{af}}^-} d_{y,[x]}\mathfrak{S}_y \in H_*^T(\mathcal{G}/\mathcal{P}_0),$$

noting that the summation contains only finitely many nonzero terms.

(iii) Denote $c'_{x,y} = c_{x,y}\big|_{\alpha_0=-\theta}$. It follows from (ii) and Proposition 3.2 that

$$\mathfrak{S}_x = \sum_{y\in W_{\text{af}}} \delta_{x,y}\mathfrak{S}_y = \sum_{y\in W_{\text{af}}}\sum_{z\in W_{\text{af}}} c'_{x,z}d_{y,[z]}\mathfrak{S}_y = \sum_{z\in W_{\text{af}}} c'_{x,z}\psi_z.$$

Note that $\psi_z = \psi_y$ whenever $z \in yW$, and each coset yW has a unique representative of translation in $Q^\vee \cong W_{\text{af}}/W$. Hence for any $x \in W_{\text{af}}^-$,

$$\mathfrak{S}_x = \sum_{z\in W_{\text{af}}} c'_{x,z}\psi_z = \sum_{t\in Q^\vee}\big(\sum_{z\in tW} c'_{x,z}\big)\psi_t = \sum_{t\in Q^\vee} c_{x,[t]}\psi_t.$$

□

The above proposition was essentially contained in Peterson's notes [35].

The Pontryagin product gives an associative S-algebra structure on the equivariant homology $H_*^T(\Omega K)$. The following proposition was stated in [35] by Peterson. We learned the following proof from Thomas Lam.

Proposition 3.7. *For any* $\lambda, \mu \in Q^\vee$, *the Pontryagin product of* ψ_{t_λ} *and* ψ_{t_μ} *in* $H_*^T(\Omega K)$ *is given by* $\psi_{t_\lambda}\psi_{t_\mu} = \psi_{t_\lambda t_\mu} = \psi_{t_{\lambda+\mu}}$.

Proof. Identify $\lambda \in Q^\vee$ with the co-character $\lambda : \mathbb{S}^1 \to T$, which gives a point $\lambda : \mathbb{S}^1 \to K$ in ΩK. These are the T-fixed points of ΩK. Note that ψ_t is the map

$H_T^*(\Omega K) \to H_T^*(\mathrm{pt})$ induced by the map $\mathrm{pt} \to \Omega K$ with image t. Thus $\psi_{t_\lambda}\psi_{t_\mu}$ is the map $H_T^*(\Omega K) \longrightarrow H_T^*(\mathrm{pt})$ induced by the following composition of maps:

$$\mathrm{pt} \longrightarrow \Omega K \times \Omega K \longrightarrow \Omega K, \quad \text{which is given by } \mathrm{pt} \mapsto (t_\lambda, t_\mu) \mapsto t_\lambda t_\mu.$$

Note that pointwise multiplication on the group takes the loops $\lambda : \mathbb{S}^1 \to T$, $\mu : \mathbb{S}^1 \to T$ to the loop $(\lambda + \mu) : \mathbb{S}^1 \to T$. Thus $\psi_{t_\lambda}\psi_{t_\mu} = \psi_{t_\lambda t_\mu} = \psi_{t_{\lambda+\mu}}$.

□

Now we can derive the proof of Theorem 3.3 easily.

Proof of Theorem 3.3. It follows from Proposition 3.6 and Proposition 3.7 that

$$\begin{aligned}
\mathfrak{S}_x\mathfrak{S}_y &= \Big(\sum_{\lambda \in Q^\vee} c_{x,[t_\lambda]}\psi_{t_\lambda}\Big)\Big(\sum_{\mu \in Q^\vee} c_{y,[t_\mu]}\psi_{t_\mu}\Big) \\
&= \textstyle\sum_{\lambda,\mu \in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}\psi_{t_\lambda}\psi_{t_\mu} \\
&= \textstyle\sum_{\lambda,\mu \in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}\psi_{t_{\lambda+\mu}} \\
&= \sum_{\lambda,\mu \in Q^\vee, z \in W_{\mathrm{af}}^-} c_{x,[t_\lambda]}c_{y,[t_\mu]}d_{z,[t_{\lambda+\mu}]}\mathfrak{S}_z.
\end{aligned}$$

Hence, $b_{x,y}^z = \sum_{\lambda,\mu \in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}d_{z,[t_{\lambda+\mu}]}$. □

Remark 3.8. *The equivariant Schubert structure constants for the equivariant cohomology of based loop group ΩK can also be expressed in terms of $c_{x,[y]}$ and $d_{x,[y]}$. (See appendix 7.4 for more details.)*

Note that $c_{x,[t_\lambda]}, c_{y,[t_\mu]}$ and $d_{z,[t_{\lambda+\mu}]}$ are homogeneous rational functions of degree $-\ell(x), -\ell(y)$ and $\ell(z)$ respectively. Since $b_{x,y}^z$ is a polynomial, we obtain the following corollary.

Corollary 3.9. *Let $x, y, z \in W_{\mathrm{af}}^-$, one has $b_{x,y}^z = 0$ unless $\ell(z) \geq \ell(x) + \ell(y)$. Furthermore if $\ell(z) = \ell(x) + \ell(y)$, then the rational function $b_{x,y}^z$ is a constant.*

4. Calculations for structure coefficients

In this section, we analyze the summation in the formula for the structure coefficients $b_{x,y}^z$'s. We obtained several useful formulas, including the following two as the main results of this section. The proofs are elementary and combinatorial in nature. These two formulas have been obtained by Lam and Shimozono [26] by a different method using nil-Hecke ring and Peterson j-isomorphism.

Proposition 4.1. *For any $wt_\lambda, t_\mu \in W_{\mathrm{af}}^-$, one has $\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu} = \mathfrak{S}_{wt_{\lambda+\mu}}$.*

Proposition 4.2. *Let $\sigma_i t_\lambda, ut_\mu \in W_{\mathrm{af}}^-$, where $\sigma_i = \sigma_{\alpha_i}$ with $i \in I$. Then one has*

$$\mathfrak{S}_{\sigma_i t_\lambda}\mathfrak{S}_{ut_\mu} = (u(w_i) - w_i)\mathfrak{S}_{ut_{\lambda+\mu}} + \sum_{\gamma \in \Gamma_1}\langle \gamma^\vee, w_i\rangle \mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}} + \sum_{\gamma \in \Gamma_2}\langle \gamma^\vee, w_i\rangle \mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}}{}^{5},$$

where Γ_1 and Γ_2 are as defined in section 2.1.

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[5] The coefficient $\langle \gamma^\vee, w_i\rangle$ is always equal to zero if either $\gamma \in \Gamma_1$ and $u\sigma_\gamma t_{\lambda+\mu} \notin W_{\mathrm{af}}^-$ or $\gamma \in \Gamma_2$ and $u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee} \notin W_{\mathrm{af}}^-$.

In section 4.1, we introduce some well-known facts on the Coxeter group $W_{\rm af}$. We also prove propositions in section 4.1 as preliminaries for later simplifications. In section 4.2 and section 4.3, we simplify the summations involved in formulas for $\mathfrak{S}_{wt_\lambda}\cdot\mathfrak{S}_{t_\mu}$ and $\mathfrak{S}_{\sigma_i t_\lambda}\cdot\mathfrak{S}_{ut_\mu}$ respectively, where the main lemmas are Proposition 4.15, Proposition 4.19 and Proposition 4.20. Since the proofs are elementary, we treat them in the appendix. After computing certain $c_{x,y}$ and $d_{x,y}$, we obtain Proposition 4.1 and Proposition 4.2 in section 4.4. Finally in section 4.5, we give a useful simplified formula for general $\mathfrak{S}_x\mathfrak{S}_y$.

4.1. **Preliminaries.** Recall that $\mathcal{S}=\{\sigma_i\mid i\in I_{\rm af}\}$. Denote $\mathcal{T}=\{x\sigma_i x^{-1}\mid x\in W_{\rm af},\sigma_i\in\mathcal{S}\}=\{\sigma_\gamma\mid\gamma\in R^+_{\rm re}\}$. Let $x,x'\in W_{\rm af}$. We say x covers x', denoted by $x'\to x$ or $x'\overset{\sigma_\gamma}{\to}x$, if and only if there exists some $\sigma_\gamma\in\mathcal{T}$ such that $x=\sigma_\gamma x'$ and $\ell(x)=\ell(x')+1$. We say $x'\preccurlyeq x$ with respect to the Bruhat order $(W_{\rm af},\preccurlyeq)$ if and only if there exists a chain $x'=x_1\to x_2\to\cdots\to x_k=x$. We list some well-known facts (from [14] and [16]) for the Coxeter system $(W_{\rm af},\mathcal{S})$ as follows.

Lemma 4.3. *Let $x,y\in W_{\rm af}$ with $x=[\sigma_{\beta_1}\cdots\sigma_{\beta_r}]_{\rm red}$. Denote $\gamma_k=\sigma_{\beta_1}\cdots\sigma_{\beta_{k-1}}(\beta_k)$.*

- (a) *If $y\to x$, then there exists a unique $j, 1\leq j\leq r$, such that $x=\sigma_{\gamma_j}y$ and $y=[\sigma_{\beta_1}\cdots\sigma_{\beta_{j-1}}\sigma_{\beta_{j+1}}\cdots\sigma_{\beta_r}]_{red}$.*
- (b) *If $y\preccurlyeq x$, then $y=[\sigma_{\beta_{k_1}}\cdots\sigma_{\beta_{k_s}}]_{\rm red}$ for some subsequence $(k_1,\cdots,k_s)$, which we call an induced reduced decomposition of y from $x=[\sigma_{\beta_1}\cdots\sigma_{\beta_r}]_{\rm red}$. In particular, if $y\preccurlyeq x$ and $\ell(x)=\ell(y)+1$, then $y\overset{\sigma_{\gamma_j}}{\to}x$ for a unique j.*
- (c) (Lifting Property) *Let $\sigma_i\in\mathcal{S}$. Suppose $\ell(\sigma_i x)>\ell(x)$ and $\ell(\sigma_i y)<\ell(y)$, then the following are equivalent:*
$$\text{(i) } \sigma_i x\preccurlyeq y;\qquad \text{(ii) } x\preccurlyeq y;\qquad \text{(iii) } x\preccurlyeq\sigma_i y.$$
- (d) $\{\gamma_1,\cdots,\gamma_r\}=\{\gamma\in R^+_{\rm re}\mid x^{-1}(\gamma)\in -R^+_{\rm re}\}$; $\qquad \ell(xy)\leq\ell(x)+\ell(y)$.
- (e) *Let $\gamma\in R^+_{\rm re}$. $\ell(\sigma_\gamma x)\leq\ell(x)\Longleftrightarrow x^{-1}(\gamma)\in -R^+_{\rm re}$.*

Each element $x\in W_{\rm af}=W\ltimes Q^\vee$ can be written as $x=wt_\lambda$ for unique $w\in W$ and $t_\lambda\in Q^\vee$. Recall that the set of anti-dominant elements in $Q^\vee$ is
$$\tilde{Q}^\vee=\{\mu\in Q^\vee\mid\langle\mu,\alpha_i\rangle\leq 0,\ i\in I\}.$$
The length-minimizing representatives in $W^-_{\rm af}$ are characterized as follows.

Lemma 4.4 (see e.g. [26])**.** *Let $\lambda\in Q^\vee$ and $w\in W$.*

- (a) *If $w(\lambda)\in\tilde{Q}^\vee$, then $\ell(t_\lambda)=\langle w(\lambda),-2\rho\rangle$.*
- (b) *$wt_\lambda\in W^-_{\rm af}\Longleftrightarrow\lambda\in\tilde{Q}^\vee$, and if $\langle\lambda,\alpha_i\rangle=0$ then $w(\alpha_i)\succcurlyeq 0$. Furthermore if $wt_\lambda\in W^-_{\rm af}$, then $\ell(wt_\lambda)=\ell(t_\lambda)-\ell(w)=\langle\lambda,-2\rho\rangle-\ell(w)$.*

Corollary 4.5. *Let $\lambda\in\tilde{Q}^\vee$. Suppose λ is regular; that is, $\langle\lambda,\alpha_i\rangle\neq 0$ for each $i\in I$. Then for any $w,u\in W$, one has* (i) *$wt_\lambda\preccurlyeq t_\lambda$;* (ii) *$wt_\lambda u\preccurlyeq t_\lambda$ implies $u=1$.*

Proof. Write $w=[\sigma_{\beta_1}\cdots\sigma_{\beta_r}]_{\rm red}$. Denote $x_j=\sigma_{\beta_j}\cdots\sigma_{\beta_r}t_\lambda$ for each $1\leq j\leq r$. Since $\lambda\in\tilde{Q}^\vee$ is regular, $x_j\in W^-_{\rm af}$ for each $1\leq j\leq r+1$, where we denote $x_{r+1}=t_\lambda$. (i) $\ell(x_{j+1})=\ell(t_\lambda)-\ell(\sigma_{\beta_{j+1}}\cdots\sigma_{\beta_r})=\ell(t_\lambda)-(r-j)$. Note that $x_{j+1}=\sigma_{\beta_j}x_j$ and $\ell(x_{j+1})=\ell(x_j)+1$. Thus $x_j\preccurlyeq x_{j+1}$ for $1\leq j\leq r$. Hence, $wt_\lambda=x_1\preccurlyeq x_{r+1}=t_\lambda$. (ii) Note that $x_j\in W^-_{\rm af}$ for $1\leq j\leq r+1$. Hence, $\ell(x_j u)=\ell(x_j)+\ell(u)$, $\ell(\sigma_{\beta_j}x_j u)=\ell(x_{j+1}u)>\ell(x_j u)$ and $\ell(\sigma_{\beta_j}x_{r+1})=\ell(x_{r+1})-1<\ell(x_{r+1})$. Therefore if $x_j u\preccurlyeq x_{r+1}$, then $x_{j+1}u=\sigma_{\beta_j}x_j u\preccurlyeq x_{r+1}$ by Lemma 4.3. Hence, $x_1 u=wt_\lambda u\preccurlyeq t_\lambda=x_{r+1}$ implies $t_\lambda u=x_{r+1}u\preccurlyeq x_{r+1}$, by induction on j. Thus $\ell(u)=0$ and $u=1$. $\square$

The following useful lemma will be applied frequently later.

Lemma 4.6 (see e.g. [32])**.** *For any $\gamma \in R^+$, $\ell(\sigma_\gamma) \leq \langle \gamma^\vee, 2\rho\rangle - 1$.*

The following lemma is on the property of the longest element ω_0 in W.

Lemma 4.7. $\omega_0(-\theta) = \theta$.

Proof. (We learn the proof from Victor Reiner.) The highest root $\theta_0 \in R^+$ is characterized among all the positive roots by the property that $\langle \theta_0, \alpha_i^\vee\rangle \geq 0$ for all α_i's. Note that $\omega_0(-\theta)$ is a positive roots. Furthermore for any simple root α_i, $\ell(\sigma_{\omega_0(\alpha_i)}) = \ell(\omega_0\sigma_i\omega_0) = \ell(\omega_0) - \ell(\sigma_i\omega_0) = \ell(\omega_0) - (\ell(\omega_0) - \ell(\sigma_i)) = 1$, which implies that $\omega_0(\sigma_i) = -\sigma_j$ for some $j \in I$. Note that $\omega_0 = \omega_0^{-1}$. Hence for $\theta_0 = \omega_0(-\theta)$, $\langle \theta_0, \alpha_i^\vee\rangle = \langle \omega_0(-\theta), \alpha_i^\vee\rangle = \langle \theta, -\omega_0(\alpha_i)^\vee\rangle = \langle \theta, \alpha_j^\vee\rangle \geq 0$. Thus $\theta_0 = \theta$. $\square$

The following result was mentioned explicitly by Lusztig in section 2 of [30]. It is proved by Stembridge as a special case of Theorem 4.10 of [37].

Proposition 4.8. *For $\lambda, \mu \in \tilde{Q}^\vee$, $t_\mu \preccurlyeq t_\lambda \Longleftrightarrow \lambda \preccurlyeq \mu$; that is, $\mu - \lambda = \sum_{i\in I} a_i\alpha_i^\vee$ with $a_i \geq 0$ for each $i \in I$.*

Corollary 4.9. *Let $t_\lambda, wt_\mu \in W_{\mathrm{af}}^-$. Then $wt_\mu \preccurlyeq t_\lambda \Longleftrightarrow \lambda \preccurlyeq \mu$.*

Proof. We can assume λ, μ to be regular. (Otherwise, we take any regular $\tau \in \tilde{Q}^\vee$ and consider $\lambda + \tau, \mu + \tau$.) We claim that $wt_\mu \preccurlyeq t_\lambda \Longleftrightarrow t_\mu \preccurlyeq t_\lambda$. Hence, the statement follows from Proposition 4.8 immediately.

Indeed, we have $wt_\mu \preccurlyeq t_\mu$ by Corollary 4.5. Suppose $t_\mu \preccurlyeq t_\lambda$, then we have $wt_\mu \preccurlyeq t_\lambda$. Suppose $wt_\mu \preccurlyeq t_\lambda$. Write $w = [\sigma_{\beta_1}\cdots\sigma_{\beta_k}]_{\mathrm{red}}$. Note that $\ell(\sigma_{\beta_1}wt_\mu) = \ell(t_\mu) - \ell(w) + 1 > \ell(wt_\mu)$ and $\ell(\sigma_{\beta_1}t_\lambda) = \ell(t_\lambda) - 1 < \ell(t_\lambda)$. Hence, $\tilde{w}t_\mu \preccurlyeq t_\lambda$ holds by Lemma 4.3, where $\tilde{w} = \sigma_{\beta_1}w$ with $\ell(\tilde{w}) = \ell(w) - 1$. Thus we can deduce $t_\mu \preccurlyeq t_\lambda$ by induction on $\ell(w)$. $\square$

Lemma 4.10. *Suppose $\lambda \in Q^\vee$ satisfies $\langle \lambda, \alpha_i\rangle \leq -2$ for each $i \in I$.Then $t_\lambda \in W_{\mathrm{af}}^-$ admits a reduced decomposition of the form $t_\lambda = \omega_0\sigma_0u_1\sigma_0\cdots u_r\sigma_0$, where $u_j \in W$ for all j.*

Proof. It follows from Lemma 4.7 that $t_\lambda = t_{-\theta^\vee}t_{\lambda+\theta^\vee} = \omega_0\sigma_\theta t_{-\theta^\vee}\sigma_\theta\omega_0 t_{\lambda+\theta^\vee}$. Note that for each $i \in I$, $\langle \lambda+\theta^\vee, \alpha_i\rangle \leq -2 + \langle \theta^\vee, \alpha_i\rangle \leq -2+1 = -1$. Hence, $\lambda+\theta^\vee \in \tilde{Q}^\vee$ is regular and we have $\sigma_\theta\omega_0 t_{\lambda+\theta^\vee} \in W_{\mathrm{af}}^-$. Hence, any reduced decomposition of $\sigma_\theta\omega_0 t_{\lambda+\theta^\vee}$ must be of the form $u_1\sigma_0\cdots u_r\sigma_0$. Furthermore,

$$\begin{aligned}
\ell(t_\lambda) &\leq \ell(\omega_0\sigma_\theta t_{-\theta^\vee}) + \ell(\sigma_\theta\omega_0 t_{\lambda+\theta^\vee})\\
&= \ell(\omega_0\sigma_0) + \ell(t_{\lambda+\theta^\vee}) - \ell(\sigma_\theta\omega_0)\\
&= \ell(\omega_0) + 1 + \langle \lambda+\theta^\vee, -2\rho\rangle - (\ell(\omega_0) - \ell(\sigma_\theta))\\
&= 1 + \ell(t_\lambda) - \langle \theta^\vee, 2\rho\rangle + \ell(\sigma_\theta)\\
&\leq 1 + \ell(t_\lambda) - \langle \theta^\vee, 2\rho\rangle + \langle \theta^\vee, 2\rho\rangle - 1 = \ell(t_\lambda).
\end{aligned}$$

Hence, all inequalities are indeed equalities. Thus t_λ admits a reduced decomposition of the form $\omega_0\sigma_0u_1\sigma_0\cdots u_r\sigma_0$. $\square$

Lemma 4.11. *Suppose $\lambda \in Q^\vee$ satisfies $\langle \lambda, \alpha_i\rangle \leq -2$ for each $i \in I$. Take a reduced decomposition of the form $t_\lambda = \omega_0\sigma_0u_1\sigma_0\cdots u_r\sigma_0$. Then for any $w \in W$, the induced reduced decomposition(s) of $wt_\lambda \preccurlyeq t_\lambda$ must be of the form $u_0\sigma_0u_1\sigma_0\cdots u_r\sigma_0$ with $u_0 = w\omega_0 \in W$.*

Proof. The induced decomposition(s) of wt_λ is of the form $wt_\lambda = u_0'\sigma_0 u_1'\sigma_0\cdots u_r'\sigma_0$ with $u_0' \preccurlyeq \omega_0$ and $u_j' \preccurlyeq u_j$ for each $1\leq j\leq r$. If $u_j'\neq u_j$ for some $1\leq j\leq r$, then $\ell(u_j')\leq \ell(u_j)-1$. Note that $t_\lambda=\omega_0\sigma_0 u_1\sigma_0\cdots u_r\sigma_0 = w^{-1}u_0'\sigma_0 u_1'\sigma_0\cdots u_r'\sigma_0$. Thus

$$\begin{aligned}
\ell(\omega_0)+r+1+\sum\nolimits_{1\leq k\leq r}\ell(u_k) &= \ell(\omega_0\sigma_0u_1\sigma_0\cdots u_r\sigma_0)\\
&= \ell(w^{-1}u_0'\sigma_0u_1'\sigma_0\cdots u_r'\sigma_0)\\
&\leq \ell(w^{-1}u_0')+r+1+\sum\nolimits_{1\leq k\leq r}\ell(u_k')\\
&\leq \ell(\omega_0)+r+1+\ell(u_j)-1+\sum\nolimits_{k\neq j}\ell(u_k)\\
&= \ell(\omega_0)+r+\sum\nolimits_{1\leq k\leq r}\ell(u_k).
\end{aligned}$$

This is a contradiction and therefore the induced reduced decomposition(s) of wt_λ must be of the form $u_0\sigma_0u_1\sigma_0\cdots u_r\sigma_0$, which implies $u_0=w\omega_0$. □

4.2. **Simplification for $\mathfrak{S}_{wt_\lambda}\cdot\mathfrak{S}_{t_\mu}$.** For any $x,y\in W_{\rm af}$, we denote $c'_{x,y}=c_{x,y}\big|_{\alpha_0=-\theta}$. The main result of this subsection is the following proposition, which says that the summation for the product contains at most one nonzero term when one of the Schubert classes is defined by a translation.

Proposition 4.12. *Let $wt_\lambda, t_\mu\in W_{\rm af}^-$. If $\langle\mu,\alpha_i\rangle < -\ell(\omega_0)$ for all $i\in I$, then*

$$\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu}=c'_{wt_\lambda,wt_\lambda}c'_{t_\mu,wt_\mu}d_{wt_{\lambda+\mu},[wt_{\lambda+\mu}w^{-1}]}\mathfrak{S}_{wt_{\lambda+\mu}}.$$

In order to prove this proposition, we first need a few lemmas.

Denote $[z]$ the coset zW for $z\in W_{\rm af}$. Each coset $[z]$ contains a unique element $m_{[z]}\in W_{\rm af}^-$ of minimum length and a unique element of translation $t_{[z]}\in Q^\vee$. Note that if $m_{[z]}=v_1t_{\lambda_1}$, then $t_{[z]}=t_{v_1(\lambda_1)}=v_1t_{\lambda_1}v_1^{-1}$.

Definition 4.13. *The length of a coset $[z]$ is defined to be $\ell([z])=\ell(m_{[z]})$. Let $x\in W_{\rm af}^-$. We define* (i) *$x\preccurlyeq[z]$ if $x\preccurlyeq m_{[z]}$ and* (ii) *$[z]\preccurlyeq x$ if $m_{[z]}\preccurlyeq x$.*

Lemma 4.14. *Let $x=wt_\lambda\in W_{\rm af}^-, z\in W_{\rm af}$. If $x\preccurlyeq y$ with $y\in[z]$, then $x\preccurlyeq[z]$.*

Proof. Write $m_{[z]}=ut_\mu\in W_{\rm af}^-$, then $y=ut_\mu v$ for some $v\in W$. Let $v=[\sigma_{\beta_1}\cdots\sigma_{\beta_k}]_{\rm red}$ and denote $\tilde y=ut_\mu\sigma_{\beta_1}\cdots\sigma_{\beta_{k-1}}$. Note that $\ell(\sigma_{\beta_k}x^{-1})=\ell(x^{-1})+1>\ell(x^{-1})$ and $\ell(\sigma_{\beta_k}y^{-1})=\ell(y^{-1})-1<\ell(y^{-1})$. Since $x\preccurlyeq y$, $x^{-1}\preccurlyeq y^{-1}$ and therefore $x^{-1}\preccurlyeq\sigma_{\beta_k}y^{-1}=\tilde y^{-1}$ by (c) of Lemma 4.3. Hence, $x\preccurlyeq\tilde y$ with $\ell(\tilde y)=\ell(y)-1$.

Hence, the statement holds by induction on $\ell(v)$. □

Proposition 4.15. *Let $x,y,z\in W_{\rm af}^-$ with $x=wt_\lambda$ and $y=t_\mu$. Let $t_1,t_2\in Q^\vee$ and denote $v_jt_{\lambda_j}=m_{[t_j]}$, j=1, 2. Suppose $x\succcurlyeq[t_1], y\succcurlyeq[t_2]$, $z\preccurlyeq[t_1t_2]$ and $\ell(z)\geq\ell(x)+\ell(y)$. If $\langle\mu,\alpha_i\rangle<-\ell(\omega_0)$ for all $i\in I$, then*

$$v_1t_{\lambda_1}=wt_\lambda,\quad v_2t_{\lambda_2}=wt_\mu\quad\textit{and}\quad z=wt_{\lambda+\mu}.$$

Proof. See appendix 7.1. □

Lemma 4.16. *Let $x,y\in W_{\rm af}^-$ with $x=wt_\lambda$ and $y=ut_\mu$. Then $c_{x,[y]}=c'_{x,y}$, if either $\lambda=\mu$ with λ regular or $\ell(x)=\ell(y)+1$.*

Proof. Recall that $c_{x,[y]}=\sum_{z\in yW}c'_{x,z}$. Given $z\in yW$, one has $z=ut_\lambda v$ for some $v\in W$. Note that $c'_{x,z}\neq0$ only if $z\preccurlyeq x$. Suppose $\lambda=\mu$ and λ is regular, then $ut_\lambda v\preccurlyeq wt_\lambda\preccurlyeq t_\lambda$ implies $v=1$ by Corollary 4.5. Suppose $\ell(x)=\ell(y)+1$. If $z\neq y$,

then $\ell(z) > \ell(y)$ and therefore $z = x$ follows from $z \preccurlyeq x$, which contradicts to the uniqueness of the length-minimizing element in each coset.

Hence, $c_{x,[y]} = c'_{x,y}$ if either of the two assumptions holds. □

Proof of Proposition 4.12. By Corollary 3.9, $\mathfrak{S}_x\mathfrak{S}_y = \sum_z b^z_{x,y}\mathfrak{S}_z$ with summation running over those $z \in W^-_{\rm af}$ satisfying $\ell(z) \geq \ell(x)+\ell(y)$ and the structure coefficient $b^z_{x,y} = \sum_{t_1,t_2} c_{x,[t_1]}c_{y,[t_2]}d_{z,[t_1t_2]}$. Here $c_{x,[t_1]} = 0$ unless $x \succcurlyeq [t_1]$; $c_{y,[t_2]} = 0$ unless $y \succcurlyeq [t_2]$; $d_{z,[t_1t_2]} = 0$ unless $z \preccurlyeq t_1t_2$, which implies $z \preccurlyeq [t_1t_2]$ by Lemma 4.14. Hence, our result follows from Proposition 4.15 and Lemma 4.16 immediately. □

4.3. **Simplification for $\mathfrak{S}_{\sigma_it_\lambda} \cdot \mathfrak{S}_{ut_\mu}$.** Recall that for $u \in W$,

$$\Gamma_1 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u)+1\}, \Gamma_2 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u)+1-\langle\gamma^\vee, 2\rho\rangle\}.$$

The main result of this subsection is as follows.

Proposition 4.17. *Let $x, y \in W^-_{\rm af}$ with $x = \sigma_it_\lambda$ and $y = ut_\mu$, where $\sigma_i = \sigma_{\alpha_i}$ for some $i \in I$. Suppose $\langle\lambda, \alpha_j\rangle < -\ell(\omega_0)$ and $\langle\mu, \alpha_j\rangle < -\ell(\omega_0)$ for all $j \in I$, then*

$$\mathfrak{S}_x\mathfrak{S}_y = c'_{x,ut_\lambda}c'_{y,y}d_{ut_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]}\mathfrak{S}_{ut_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_1} A_\gamma\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_2} B_\gamma\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}},$$

$$\begin{aligned} \textit{where}\quad A_\gamma &= c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]} + c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_\mu}d_{u\sigma_\gamma t_{\lambda+\mu},[u\sigma_\gamma t_{\lambda+\mu}\sigma_\gamma u^{-1}]},\\ B_\gamma &= c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[ut_{\lambda+\mu}u^{-1}]} +\\ &\quad c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_{\mu+\gamma^\vee}}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}\sigma_\gamma u^{-1}]}.\end{aligned}$$

Lemma 4.18. *Let $x, y, z \in W^-_{\rm af}$ with $x = \sigma_it_\lambda$ and $y = ut_\mu$ where $i \in I$. Let $t_j \in Q^\vee$ and denote $v_jt_{\lambda_j} = m_{[t_j]}$, j=1, 2. Suppose $x \succcurlyeq [t_1], y \succcurlyeq [t_2], z \preccurlyeq [t_1t_2]$ and $\ell(z) \geq \ell(x)+\ell(y)$. Then only the following two possibilities can happen,*

Case A: $\ell([t_1t_2]) = \ell(x)+\ell(y)+1$; **Case B:** $\ell([t_1t_2]) = \ell(z) = \ell(x)+\ell(y)$.

Proof. Note that $z \preccurlyeq [t_1t_2]$ implies $\ell(z) \leq \ell([t_1t_2])$. Therefore,

$$\begin{aligned}\ell(x)+\ell(y) \leq \ell([t_1t_2]) &= \ell([v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}])\\ &\leq \ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\ &\leq \ell(v_1t_{\lambda_1}) + \ell(v_1^{-1}) + \ell(v_2t_{\lambda_2})\\ &= \ell(t_{\lambda_1}) + \ell(v_2t_{\lambda_2})\\ &\leq \ell(t_\lambda) + \ell(y) = \ell(x)+1+\ell(y).\end{aligned}$$

Hence, only two cases (**Case A** or **Case B**) are possible. □

Proposition 4.19. *Under the same assumptions as in Lemma 4.18, we assume λ is regular. If* **Case A** *occurs, then $v_1t_{\lambda_1} = ut_\lambda$ and $v_2t_{\lambda_2} = ut_\mu$. Furthermore, only one of the following three possibilities can happen,*

a) $z = ut_{\lambda+\mu}$;
b) *there exists $\gamma \in \Gamma_1$ such that $z = u\sigma_\gamma t_{\lambda+\mu}$;*
c) *there exists $\gamma \in \Gamma_2$ such that $z = u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}$.*

Proof. See appendix 7.2. □

Proposition 4.20. *Under the same assumptions as in Lemma 4.18, we assume that* $\langle\lambda,\alpha_j\rangle < -\ell(\omega_0)$ *and* $\langle\mu,\alpha_j\rangle < -\ell(\omega_0)$ *for all* $j\in I$. *If* **Case B** *occurs, then only one of the following two possibilities can happen,*
a) *there exists* $\gamma\in\Gamma_1$ *such that* $v_1t_{\lambda_1}=u\sigma_\gamma t_\lambda$, $v_2t_{\lambda_2}=u\sigma_\gamma t_\mu$, $z=u\sigma_\gamma t_{\lambda+\mu}$*;*
b) *there exists* $\gamma\in\Gamma_2$ *such that* $v_1t_{\lambda_1}=u\sigma_\gamma t_\lambda$, $v_2t_{\lambda_2}=u\sigma_\gamma t_{\mu+\gamma^\vee}$, $z=u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}$.

Proof. See appendix 7.3. □

Proof of Proposition 4.17. Note that $\mathfrak{S}_x\mathfrak{S}_y=\sum_{z\in W_{\rm af}^-}\sum_{t_1,t_2}c_{x,[t_1]}c_{y,[t_2]}d_{z,[t_1t_2]}\mathfrak{S}_z$, where the only nonzero terms are those satisfying $x\succcurlyeq[t_1], y\succcurlyeq[t_2], z\preccurlyeq[t_1t_2]$ and $\ell(z)\geq\ell(x)+\ell(y)$. Therefore, our result follows from Proposition 4.19, Proposition 4.20 and Lemma 4.16 immediately. □

4.4. **Calculations for** $\mathfrak{S}_{wt_\lambda}\cdot\mathfrak{S}_{t_\mu}$ **and** $\mathfrak{S}_{\sigma_it_\lambda}\cdot\mathfrak{S}_{ut_\mu}$**.** The main results of this subsection are Proposition 4.1 and Proposition 4.2 as stated in the beginning of this section.

The reduced decomposition(s) of $w\in W\subset W_{\rm af}$ does not contain the simple reflection σ_{α_0}. That is, reduced decompositions of $w\in W$ with respect to $(W_{\rm af},\mathcal{S})$ are exactly the same as reduced decompositions of w with respect to the Coxeter sub-system $(W,\{\sigma_{\alpha_j}\mid j\in I\})$. As a consequence, $d_{u,v}\in\mathbb{Q}[\alpha_1,\cdots,\alpha_n]$ and $c_{u,v}\in\mathbb{Q}[\alpha_1^\pm,\cdots,\alpha_n^\pm]$ for any $u,v\in W$.

The following lemma is useful for calculations of certain structure constants.

Lemma 4.21 (Lemma 11.1.22 of [24])**.** *For any* $v,u\in W$,

$$d_{v,u}=\Big(\prod_{\beta\in R^+}\beta\Big)u\big(c_{v^{-1}\omega_0,u^{-1}\omega_0}\big).$$

Lemma 4.22. *For any* $x,y\in W_{\rm af}^-$ *and any* $w\in W$, *one has* $d_{x,yw}=d_{x,y}$.

Proof. Let $y=[\sigma_{\beta_1}\cdots\sigma_{\beta_k}]_{\rm red}$ and $w=[\sigma_{\beta_{k+1}}\cdots\sigma_{\beta_{k+s}}]_{\rm red}$. Then $\beta_{k+1},\cdots,\beta_{k+s}\in\Delta$, $\beta_k=\alpha_0$ and $yw=[\sigma_{\beta_1}\cdots\sigma_{\beta_{k+s}}]_{\rm red}$. For any subsequence $J=(j_1,\cdots,j_a)$ of $(1,\cdots,k+s)$, we denote $\sigma_J=\sigma_{\beta_{j_1}}\cdots\sigma_{\beta_{j_a}}$. Suppose $x=[\sigma_J]_{\rm red}$, then J must be a subsequence of $(1,\cdots,k)$. Therefore, $d_{x,yw}=d_{x,y}$ by definition. Indeed, if $J=J_1\bigsqcup J_2$ with $J_2\subset(k+1,\cdots,k+s)$ nonempty and $J_1\subset(1,\cdots,k)$, then $\sigma_{J_2}\in W$ and $\ell(x(\sigma_{J_2})^{-1})=\ell(\sigma_{J_1}\sigma_{J_2}(\sigma_{J_2})^{-1})=\ell(\sigma_{J_1})\leq|J_1|<|J|=\ell(x)$, which contradicts to the fact that x is length-minimizing in the coset xW. □

Suppose that $\langle\lambda,\alpha_i\rangle\leq-2$ and $\langle\mu,\alpha_i\rangle\leq-2$ for all $i\in I$. Let $m=\ell(t_\lambda)$ and $p=\ell(t_\mu)$. Because of Lemma 4.10, we can take reduced expressions $t_\lambda=[\sigma_{\beta_1}\cdots\sigma_{\beta_m}]_{\rm red}$ and $t_\mu=[\sigma_{\beta_{m+1}}\cdots\sigma_{\beta_{m+p}}]_{\rm red}$ such that $\beta_{r+1}=\beta_{m+r+1}=\alpha_0$ and $\omega_0=[\sigma_{\beta_1}\cdots\sigma_{\beta_r}]_{\rm red}=[\sigma_{\beta_{m+1}}\cdots\sigma_{\beta_{m+r}}]_{\rm red}$, where $r=\ell(\omega_0)$. Note that $t_{\lambda+\mu}=[\sigma_{\beta_1}\sigma_{\beta_2}\cdots\sigma_{\beta_{m+p}}]_{\rm red}$. Denote $H_j=\sigma_{\beta_1}\cdots\sigma_{\beta_{j-1}}(\beta_j)$ for $1\leq j\leq m+p$, and denote $\tilde H_j=\sigma_{\beta_{m+1}}\cdots\sigma_{\beta_{m+j-1}}(\beta_{m+j})$ for $1\leq j\leq p$. *We will adopt the same notations of* H_j*'s and* $\tilde H_k$*'s in this subsection and section 5.3.*

Note that $H_{m+j}=t_\lambda(\tilde H_j)$ for $1\leq j\leq p$, and that for any $w\in W$ one has

$$\prod_{i=1}^r H_i=\prod_{i=1}^r\tilde H_i=\prod_{\beta\in R^+}\beta=(-1)^{\ell(w)}w\Big(\prod_{\beta\in R^+}\beta\Big)=(-1)^{\ell(w)}\prod_{\beta\in R^+}w(\beta).$$

Lemma 4.23. *Suppose that $\langle\lambda,\alpha_i\rangle \leq -2$ and $\langle\mu,\alpha_i\rangle \leq -2$ for all $i\in I$. For any $v,u,w\in W$, one has the following*

(1) $d_{v,v} = \prod_{\substack{\gamma\in R^+\\ v^{-1}(\gamma)\in -R^+}} \gamma.$

(2) $d_{vt_\lambda,vt_\lambda} = \Big(\prod_{\substack{\gamma\in R^+\\ v^{-1}(\gamma)\in R^+}} \gamma\Big)\cdot \prod_{j=r+1}^{m} v(H_j).$

(3) $c_{vt_\lambda,ut_\lambda} = \dfrac{u(d_{v^{-1},u^{-1}})}{\prod_{j=1}^{m} u(H_j)}.$

(4) $w(d_{w^{-1},w^{-1}})\cdot d_{wt_{\lambda+\mu},wt_{\lambda+\mu}} = \prod_{j=1}^{m+p} w(H_j).$

Proof. Due to Lemma 4.11, we can write $vt_\lambda = [\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}\sigma_{\beta_{r+1}}\cdots\sigma_{\beta_m}]_{\rm red}$ with $(i_1,\cdots,i_k)$ a subsequence of $(1,\cdots,r)$ and $v\omega_0 = [\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}]_{\rm red}$.

(1) Let $v = [\sigma_{\beta_{M+1}}\cdots\sigma_{\beta_{M+s}}]_{\rm red}$ and denote $\gamma_j = \sigma_{\beta_{M+1}}\cdots\sigma_{\beta_{M+j-1}}(\beta_{M+j})$, where $M > m+p$. Then $d_{v,v} = \prod_{j=1}^{s}\gamma_j = \prod_{\substack{\gamma\in R^+\\ v^{-1}(\gamma)\in -R^+}}\gamma$, by Lemma 4.3.

(2) $d_{vt_\lambda,vt_\lambda} = \Big(\prod_{j=1}^{k}\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_{j-1}}}(\beta_{i_j})\Big)\cdot \prod_{j=r+1}^{m}\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}\sigma_{\beta_{r+1}}\cdots\sigma_{\beta_{j-1}}(\beta_j)$ $= d_{v\omega_0,v\omega_0}\cdot \prod_{j=r+1}^{m} v(H_j) = \Big(\prod_{\substack{\gamma\in R^+\\ \omega_0 v^{-1}(\gamma)\in -R^+}}\gamma\Big)\cdot \prod_{j=r+1}^{m} v(H_j)$. Since ω_0 is an involution that maps $-R^+$ to R^+, the statement holds.

(3) It follows from Lemma 4.11 that $(\varepsilon_{i_1},\cdots,\varepsilon_{i_k},\varepsilon_{r+1},\cdots,\varepsilon_m)\in\{0,1\}^{m-r+k}$ satisfies $\sigma_{\beta_{i_1}}^{\varepsilon_{i_1}}\cdots\sigma_{\beta_{i_k}}^{\varepsilon_{i_k}}\sigma_{\beta_{r+1}}^{\varepsilon_{r+1}}\cdots\sigma_{\beta_m}^{\varepsilon_m} = ut_\lambda$ only if $\varepsilon_{r+1}=\cdots=\varepsilon_m=1$ and $\sigma_{\beta_{i_1}}^{\varepsilon_{i_1}}\cdots\sigma_{\beta_{i_k}}^{\varepsilon_{i_k}} = u\omega_0$. As a consequence, one has $c_{vt_\lambda,ut_\lambda} = c_{v\omega_0,u\omega_0}\cdot \frac{1}{\prod_{j=r+1}^{m}u(H_j)}$ by definition. Note that $\prod_{i=1}^{r}H_i = \prod_{\beta\in R^+}\beta$. Then the statement follows from Lemma 4.21 immediately.

(4) Note that the set $\{\gamma\in R^+ \mid w(\gamma)\in R^+\}$ is w-invariant and $\prod_{i=1}^{r}H_i = \prod_{\beta\in R^+}\beta$. Therefore, it follows from (1) and (2) that

$$\begin{aligned}
w(d_{w^{-1},w^{-1}})\cdot d_{wt_{\lambda+\mu},wt_{\lambda+\mu}} &= w\Big(\prod_{\substack{\gamma\in R^+\\ w(\gamma)\in -R^+}}\gamma\Big)\Big(\prod_{\substack{\gamma\in R^+\\ w^{-1}(\gamma)\in R^+}}\gamma\Big)\cdot\prod_{j=r+1}^{m+p} w(H_j)\\
&= w\Big(\prod_{\substack{\gamma\in R^+\\ w(\gamma)\in -R^+}}\gamma\Big)\cdot w\Big(\prod_{\substack{\gamma\in R^+\\ w(\gamma)\in R^+}}\gamma\Big)\cdot\prod_{j=r+1}^{m+p} w(H_j)\\
&= w\Big(\prod_{\gamma\in R^+}\gamma\Big)\cdot\prod_{j=r+1}^{m+p} w(H_j)\\
&= w\Big(\prod_{j=1}^{r}H_j\Big)\cdot\prod_{j=r+1}^{m+p} w(H_j) = \prod_{j=1}^{m+p} w(H_j). \qquad\square
\end{aligned}$$

Proof of Proposition 4.1. We first assume $\langle\mu,\alpha_i\rangle < -\ell(\omega_0)$ for each $i\in I$. Note that $t_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta} = \tilde H_j\big|_{\alpha_0=-\theta}$. By definition, $d_{wt_{\lambda+\mu},wt_{\lambda+\mu}} = d_{wt_\lambda,wt_\lambda}\prod_{j=1}^{p} wt_\lambda(\tilde H_j)$,

$d_{1,w^{-1}}=1$, and $c_{wt_\lambda,wt_\lambda}d_{wt_\lambda,wt_\lambda}=1$. By Lemma 4.22 and Lemma 4.23, we have

$$\begin{aligned}c'_{wt_\lambda,wt_\lambda}c'_{t_\mu,wt_\mu}d_{wt_{\lambda+\mu},[wt_{\lambda+\mu}w^{-1}]} =&c_{wt_\lambda,wt_\lambda}\cdot\frac{w(d_{1,w^{-1}})}{\prod_{j=1}^p w(\tilde H_j)}\cdot d_{wt_\lambda,wt_\lambda}\prod_{j=1}^p wt_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}\\ =&\frac{\prod_{j=1}^p wt_\lambda(\tilde H_j)}{\prod_{j=1}^p w(\tilde H_j)}\big|_{\alpha_0=-\theta}=1.\end{aligned}$$

Hence, it follows from Proposition 4.12 that $\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu}=\mathfrak{S}_{wt_{\lambda+\mu}}$.

In general, we take $\kappa\in\tilde Q^\vee$ such that $\langle\kappa,\alpha_i\rangle<-\ell(\omega_0)$ and $\langle\kappa+\mu,\alpha_i\rangle<-\ell(\omega_0)$ for each $i\in I$. Denote $x=wt_\lambda, y=t_\mu$. Because of the associativity of the product,

$$\mathfrak{S}_{xt_{\mu+\kappa}}=\mathfrak{S}_x\mathfrak{S}_{t_{\mu+\kappa}}=\mathfrak{S}_x(\mathfrak{S}_y\mathfrak{S}_{t_\kappa})=(\mathfrak{S}_x\mathfrak{S}_y)\mathfrak{S}_{t_\kappa}=\sum\nolimits_z b^z_{x,y}\mathfrak{S}_z\mathfrak{S}_{t_\kappa}=\sum\nolimits_z b^z_{x,y}\mathfrak{S}_{zt_\kappa}.$$

Hence, $b^z_{x,y}=0$ if $zt_\kappa\neq xt_{\mu+\kappa}$; $b^z_{x,y}=1$ if $zt_\kappa=xt_{\mu+\kappa}$, that is, $z=wt_{\lambda+\mu}$. □

Lemma 4.24. *Suppose that $\langle\lambda,\alpha_j\rangle<-\ell(\omega_0)$ and $\langle\mu,\alpha_j\rangle<-\ell(\omega_0)$ for all $j\in I$. Let $v,u\in W$ and $\sigma_i=\sigma_{\alpha_i}$ with $i\in I$. Then one has $d_{\sigma_i,u}=w_i-u(w_i)$. Furthermore,*

(1) $d_{u\sigma_\gamma t_{\lambda+\mu},ut_{\lambda+\mu}}=\dfrac{1}{u(\gamma)}\cdot d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}$, *for any* $\gamma\in\Gamma_1$.

(2) $d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},ut_{\lambda+\mu}}=\dfrac{1}{u(\gamma+\delta)}\cdot d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}$, *for any* $\gamma\in\Gamma_2$.

(3) $c_{ut_\mu,u\sigma_\gamma t_{\mu+\gamma^\vee}}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}}=-\dfrac{\prod_{j=1}^m u\sigma_\gamma t_{\mu+\gamma^\vee}(H_j)}{u(\gamma+\delta)}$, *for any* $\gamma\in\Gamma_2$.

Proof. $d_{\sigma_i,u}=w_i-u(w_i)$ holds by expanding the right side (with respect to a reduced expression of u) and comparing both sides. Let $ut_\mu=[\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}]_{\rm red}$, $ut_{\lambda+\mu}=[\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_s}}]_{\rm red}$ $(k<s)$ and denote $\gamma_j=\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_{j-1}}}(\beta_{i_j})$.

Note that $u\sigma_\gamma t_{\mu+\gamma^\vee},u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}\in W^-_{\rm af}$ whenever $\gamma\in\Gamma_2$, by Remark 7.1. Therefore for $\gamma\in\Gamma_2$, $\ell(u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee})=\ell(ut_{\lambda+\mu})-1$ and $\ell(u\sigma_\gamma t_{\mu+\gamma^\vee})=\ell(ut_\mu)-1$.

Note that $u(\gamma+\delta)=u(\gamma)+\delta\in R^+_{\rm re}$ and $u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}=\sigma_{u(\gamma+\delta)}ut_{\lambda+\mu}$. Hence, there is a unique $1\leq j\leq s$ such that $u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}=[\sigma_{\beta_{i_1}}\cdots\widehat{\sigma_{\beta_{i_j}}}\cdots\sigma_{\beta_{i_s}}]_{\rm red}$ and $\gamma_j=u(\gamma+\delta)$ by Lemma 4.3. As a consequence, $d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},ut_{\lambda+\mu}}=\frac{1}{\gamma_j}\prod_{a=1}^s\gamma_a=\frac{1}{u(\gamma+\delta)}d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}$. Hence, (2) holds. Similarly, (1) also holds.

With the same argument as above, there is a unique $1\leq j\leq k$ such that $\gamma_j=u(\gamma+\delta)$ and $u\sigma_\gamma t_{\mu+\gamma^\vee}=\sigma_{\beta_{i_1}}\cdots\widehat{\sigma_{\beta_{i_j}}}\cdots\sigma_{\beta_{i_k}}$. Denote $(a_1,\cdots,a_{k-1})=(i_1,\cdots,\hat i_j,\cdots,i_k)$ and denote $\tilde\gamma_b=\sigma_{\beta_{a_1}}\cdots\sigma_{\beta_{a_{b-1}}}(\beta_{a_b})$. Immediately, we have $c_{ut_\mu,u\sigma_\gamma t_{\mu+\gamma^\vee}}=-(\gamma_j\prod_{b=1}^{k-1}\tilde\gamma_b)^{-1}$ and $d_{u\sigma_\gamma t_{\mu+\gamma^\vee},u\sigma_\gamma t_{\mu+\gamma^\vee}}=\prod_{b=1}^{k-1}\tilde\gamma_b$ by definition. Therefore, (3) also holds by the following observation

$$d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}}=d_{u\sigma_\gamma t_{\mu+\gamma^\vee},u\sigma_\gamma t_{\mu+\gamma^\vee}}\cdot\prod\nolimits_{j=1}^m u\sigma_\gamma t_{\mu+\gamma^\vee}(H_j).$$ □

Proof of Proposition 4.2. Denote $x=\sigma_it_\lambda$ and $y=ut_\mu$.

We first assume $\langle\lambda,\alpha_k\rangle<-\ell(\omega_0)$ and $\langle\mu,\alpha_k\rangle<-\ell(\omega_0)$ and for each $k\in I$. Note that $t_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}=\tilde H_j\big|_{\alpha_0=-\theta}$. It follows from Lemma 4.22, Lemma 4.23

and Lemma 4.24 that

$$\begin{aligned}c'_{x,ut_\lambda}c'_{y,y}d_{ut_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]} &= \frac{u(d_{\sigma_i,u^{-1}})}{\prod_{j=1}^m u(H_j)}\cdot\frac{u(d_{u^{-1},u^{-1}})}{\prod_{j=1}^p u(\tilde H_j)}\cdot d_{ut_\lambda,ut_\lambda}\prod_{j=1}^p ut_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}\\ &= \frac{u(w_i-u^{-1}(w_i))}{\prod_{j=1}^m u(H_j)}\cdot\frac{\prod_{j=1}^m u(H_j)}{\prod_{j=1}^p u(\tilde H_j)}\cdot\prod_{j=1}^p u(\tilde H_j)\big|_{\alpha_0=-\theta}\\ &= u(w_i)-w_i.\end{aligned}$$

For $\gamma\in\Gamma_1$, one has

$$\begin{aligned}&c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]}+c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_\mu}d_{u\sigma_\gamma t_{\lambda+\mu},[u\sigma_\gamma t_{\lambda+\mu}\sigma_\gamma u^{-1}]}\\ =&c_{x,ut_\lambda}c_{y,y}\frac{d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}}{u(\gamma)}+\frac{u\sigma_\gamma(d_{\sigma_i,\sigma_\gamma u^{-1}})}{\prod_{j=1}^m u\sigma_\gamma(H_j)}\cdot\frac{u\sigma_\gamma(d_{u^{-1},\sigma_\gamma u^{-1}})}{\prod_{j=1}^p u\sigma_\gamma(\tilde H_j)}d_{u\sigma_\gamma t_{\lambda+\mu},u\sigma_\gamma t_{\lambda+\mu}}\big|_{\alpha_0=-\theta}\\ =&\frac{u(w_i)-w_i}{u(\gamma)}+\frac{u\sigma_\gamma(w_i)-w_i}{\prod_{j=1}^{m+p}u\sigma_\gamma(H_j)}\cdot u\sigma_\gamma(\frac{d_{\sigma_\gamma u^{-1},\sigma_\gamma u^{-1}}}{\gamma})d_{u\sigma_\gamma t_{\lambda+\mu},u\sigma_\gamma t_{\lambda+\mu}}\big|_{\alpha_0=-\theta}\\ =&\frac{u(w_i)-w_i}{u(\gamma)}-\frac{u\sigma_\gamma(w_i)-w_i}{\prod_{j=1}^{m+p}u\sigma_\gamma(H_j)}\cdot\frac{\prod_{j=1}^{m+p}u\sigma_\gamma(H_j)}{u(\gamma)}\big|_{\alpha_0=-\theta}\\ =&\frac{u(w_i-\sigma_\gamma(w_i))}{u(\gamma)}=\langle\gamma^\vee,w_i\rangle.\end{aligned}$$

For $\gamma\in\Gamma_2$, one has

$$\begin{aligned}&c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[ut_{\lambda+\mu}u^{-1}]}+c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_{\mu+\gamma^\vee}}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}\sigma_\gamma u^{-1}]}\\ =&c_{x,ut_\lambda}c_{y,y}\frac{d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}}{u(\gamma+\delta)}+\frac{u\sigma_\gamma(d_{\sigma_i,\sigma_\gamma u^{-1}})}{\prod_{j=1}^m u\sigma_\gamma(H_j)}\cdot\frac{-\prod_{j=1}^m u\sigma_\gamma t_{\mu+\gamma^\vee}(H_j)}{u(\gamma+\delta)}\big|_{\alpha_0=-\theta}\\ =&\frac{u(w_i)-w_i}{u(\gamma)}-\frac{u\sigma_\gamma(w_i)-w_i}{\prod_{j=1}^m u\sigma_\gamma(H_j)}\cdot\frac{\prod_{j=1}^m u\sigma_\gamma(H_j)}{u(\gamma)}\big|_{\alpha_0=-\theta}\\ =&\frac{u(w_i-\sigma_\gamma(w_i))}{u(\gamma)}=\langle\gamma^\vee,w_i\rangle.\end{aligned}$$

Hence, the statement holds.

For general cases, the statement follows from Proposition 4.1 and the associativity and the commutativity of the Pontryagin product. □

4.5. **Simplification for summation in $b^z_{x,y}$.** Denote $v_it_{\lambda_i}=m_{[t_i]}\in W^-_{\mathrm{af}}$ the length-minimizing element in the coset $[t_i]=t_iW$, where $t_i\in Q^\vee, i=1,2$. Note that for any $x\in W^-_{\mathrm{af}}$ one has that $c_{x,[t_i]}=c_{x,[v_it_{\lambda_i}]}$ by definition, and that $d_{x,t_1t_2}=d_{x,v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}v_2^{-1}}=d_{x,v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}}$ following from Lemma 4.22. Therefore for any $x,y,z\in W^-_{\mathrm{af}}$,

$$b^z_{x,y}=\sum_{t_1,t_2\in Q^\vee}c_{x,[t_1]}c_{y,[t_2]}d_{z,[t_1t_2]}=\sum_{v_1t_{\lambda_1},v_2t_{\lambda_2}\in W^-_{\mathrm{af}}}c_{x,[v_1t_{\lambda_1}]}c_{y,[v_2t_{\lambda_2}]}d_{z,[v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}]}.$$

The following lemma is contained in Lemma 13.2.A of [15].

Lemma 4.25. *For any $\lambda\in Q^\vee$, there exists a unique $\lambda'\in\tilde Q^\vee$ and some $w\in W$ such that $\lambda'=w(\lambda)$. Furthermore, $\lambda'\preccurlyeq\lambda$.*

Lemma 4.26. *Let $\lambda_1, \lambda_2 \in \tilde{Q}^\vee$ with $\langle \lambda_j, \alpha_i\rangle \leq -2\ell(\omega_0)$ for all $j \in \{1, 2\}$ and $i \in I$. Let $v_1, v_2 \in W$. If $v_1 \neq v_2$, then $\ell(t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) \leq \langle \lambda_1 + \lambda_2, -2\rho\rangle - 4\ell(\omega_0)$.*

Proof. Take $w \in W$ such that $w(v_2^{-1}v_1(\lambda_1) + \lambda_2) \in \tilde{Q}^\vee$. By Lemma 4.25, one has $wv_2^{-1}v_1(\lambda_1) = \lambda_1 + \mu_1$ with $\mu_1 \succcurlyeq 0$. If $w \neq 1$, then $w = [\sigma_{\beta_1} \cdots \sigma_{\beta_k}]_{\rm red}$ with $k \geq 1$. Note that $w(\lambda_2) = \lambda_2 - \sum_{j=1}^k \langle \lambda_2, \beta_j\rangle \gamma_j^\vee$, where $\gamma_j = \sigma_{\beta_1} \cdots \sigma_{\beta_{j-1}}(\beta_j) \in R^+$. Therefore, it follows from Lemma 4.4 that

$$\begin{aligned}
\ell(t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) &= \langle \lambda_1 + \mu_1 + w(\lambda_2), -2\rho\rangle \\
&= \langle \lambda_1 + \lambda_2 + \mu_1, -2\rho\rangle + \sum\nolimits_{j=1}^k \langle \lambda_2, \beta_j\rangle\langle \gamma_j^\vee, 2\rho\rangle \\
&= \langle \lambda_1 + \lambda_2, -2\rho\rangle - \langle \mu_1, 2\rho\rangle + \langle \lambda_2, \beta_1\rangle \cdot 2 + \sum_{j=2}^k \langle \lambda_2, \beta_j\rangle\langle \gamma_j^\vee, 2\rho\rangle \\
&\leq \langle \lambda_1 + \lambda_2, -2\rho\rangle - 0 - 2\ell(\omega_0) \cdot 2 + 0.
\end{aligned}$$

If $w = 1$, then $w(v_2^{-1}v_1(\lambda_1) + \lambda_2) = \lambda_2 + w'(\lambda_1)$ with $w' = v_2^{-1}v_1 \neq 1$. With the same argument as above, one has $\ell(t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) \leq \langle \lambda_1 + \lambda_2, -2\rho\rangle - 4\ell(\omega_0)$. □

Proposition 4.27. *Let $x, y \in W_{\rm af}^-$ with $x = ut_\eta, y = vt_\kappa$. If $\langle \eta, \alpha_i\rangle \leq -5\ell(\omega_0)$ and $\langle \kappa, \alpha_i\rangle \leq -5\ell(\omega_0)$ for each $i \in I$, then one has*

$$\mathfrak{S}_x\mathfrak{S}_y = \sum_{\substack{wt_\mu \in W_{\rm af}^- \\ \ell(wt_\mu)\geq \ell(x)+\ell(y)}} \sum_{\substack{v_1\in W, \lambda_1,\lambda_2\in\tilde{Q}^\vee \\ \lambda_1\succcurlyeq\eta, \lambda_2\succcurlyeq\kappa, \lambda_1+\lambda_2\preccurlyeq\mu}} c_{x,[v_1t_{\lambda_1}]}c_{y,[v_1t_{\lambda_2}]}d_{wt_\mu,[v_1t_{\lambda_1+\lambda_2}]}\mathfrak{S}_{wt_\mu}.$$

Proof. It follows from Corollary 3.9 that $\mathfrak{S}_x\mathfrak{S}_y = \sum_{z\in W_{\rm af}^-;\ell(z)\geq\ell(x)+\ell(y)} b_{x,y}^z \mathfrak{S}_z$. Now let $z = wt_\mu \in W_{\rm af}^-$ with $\ell(z) \geq \ell(x) + \ell(y) = \langle \eta + \kappa, -2\rho\rangle - \ell(u) - \ell(v)$.

Note that $c_{x,[v_1t_{\lambda_1}]} \neq 0$ only if $v_1t_{\lambda_1} \preccurlyeq ut_\eta \preccurlyeq t_\eta$, which implies $\lambda_1 \succcurlyeq \eta$; $c_{y,[v_2t_{\lambda_2}]} \neq 0$ only if $v_2t_{\lambda_2} \preccurlyeq vt_\eta \preccurlyeq t_\kappa$, which implies $\lambda_2 \succcurlyeq \kappa$. Hence, $\lambda_1 = \eta + \lambda_3 = \eta + \sum_{i\in I} a_i\alpha_i^\vee$, $\lambda_2 = \kappa + \lambda_4 = \kappa + \sum_{i\in I} b_i\alpha_i^\vee$ with $a_i, b_i \geq 0$ for each $i \in I$. Note that $d_{z,[v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}]} \neq 0$ only if $z \preccurlyeq v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}$, in particular only if $\ell(v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) \geq \ell(z) \geq \ell(t_{\eta+\kappa}) - \ell(u) - \ell(v) \geq \ell(t_{\eta+\kappa}) - 2\ell(\omega_0)$. Furthermore,

$$\begin{aligned}
\ell(v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) &= \ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}) \\
&\leq \ell(v_1t_{\lambda_1}) + \ell(v_1^{-1}v_2) + \ell(t_{\lambda_2}) \\
&= \langle \lambda_1, -2\rho\rangle - \ell(v_1) + \ell(v_1^{-1}v_2) + \langle \lambda_2, -2\rho\rangle \\
&= \langle \eta + \kappa, -2\rho\rangle - 2\sum\nolimits_{i\in I}(a_i + b_i)\langle \alpha_i^\vee, \rho\rangle - \ell(v_1) + \ell(v_1^{-1}v_2) \\
&\leq \ell(t_{\eta+\kappa}) - 2\sum\nolimits_{i\in I}(a_i + b_i) - 0 + \ell(\omega_0).
\end{aligned}$$

Hence, $2\sum_{i\in I}(a_i + b_i) \leq 3\ell(\omega_0)$. In particular, $0 \leq 2a_i \leq 3\ell(\omega_0), 0 \leq 2b_i \leq 3\ell(\omega_0)$, $\langle \lambda_1, \alpha_i\rangle = \langle \eta, \alpha_i\rangle + a_i\langle \alpha_i^\vee, \alpha_i\rangle + \sum_{j\neq i} a_j\langle \alpha_j^\vee, \alpha_i\rangle \leq -5\ell(\omega_0) + 3\ell(\omega_0) + 0 = -2\ell(\omega_0)$ and $\langle \lambda_2, \alpha_i\rangle \leq -2\ell(\omega_0)$ for each $i \in I$. Therefore, $v_1 = v_2$; since if $v_1 \neq v_2$, then a contradiction comes out following from Lemma 4.26:

$\ell(v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) \leq \ell(v_2) + \ell(t_{\lambda_1+\lambda_2}) - 4\ell(\omega_0) \leq \ell(\omega_0) + \ell(t_{\eta+\kappa}) - 4\ell(\omega_0)$.

So far, we have shown that the effective summation for $z = wt_\mu$ runs over those elements $v_1t_{\lambda_1}, v_1t_{\lambda_2} \in W_{\rm af}^-$ with $\lambda_1 \succcurlyeq \eta$ and $\lambda_2 \succcurlyeq \kappa$. Note that $\lambda_1, \lambda_2 \in \tilde{Q}^\vee$ are

regular, then $v_1 t_{\lambda_i} \in W_{\mathrm{af}}^-$ for any $v_1 \in W$. Note that $d_{z,[v_1 t_{\lambda_1+\lambda_2}]} \neq 0$ only if $wt_\mu \preccurlyeq v_1 t_{\lambda_1+\lambda_2} \preccurlyeq t_{\lambda_1+\lambda_2}$, which implies $\lambda_1+\lambda_2 \preccurlyeq \mu$. Hence, the statement holds. $\square$

5. Equivariant quantum cohomology of G/B

In [26], Lam and Shimozono have established an equivalence between the equivariant quantum cohomology of G/B and the equivariant homology of ΩK after localization, by using Mihalcea's criterion. For completeness, we also include a proof of the equivalence. Then we obtain the main theorem of this paper as stated below, which covers Main Theorem as stated in the introduction.

Theorem 5.1. *For any $u, v, w \in W$, $\lambda \in Q^\vee$ with $\lambda \succcurlyeq 0$, the quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ for G/B is given as follows.*

Denote $A = -12n(n+1)\sum_{i\in I} w_i^\vee$ (which is in fact a regular and anti-dominant element in $Q^\vee$). Let $x = ut_A, y = vt_A$ and $z = wt_{2A+\lambda}$.

1. *If $\langle \lambda, 2\rho\rangle \neq \ell(u)+\ell(v)-\ell(w)$, then $N_{u,v}^{w,\lambda} = 0$.*
2. *If $\langle \lambda, 2\rho\rangle = \ell(u)+\ell(v)-\ell(w)$, then*
   (a) *The rational function $\sum_{\lambda_1,\lambda_2\in Q^\vee} c_{x,[t_{\lambda_1}]} c_{y,[t_{\lambda_2}]} d_{z,[t_{\lambda_1+\lambda_2}]}$, which belongs to $\mathbb{Q}[\alpha_1^\pm, \cdots, \alpha_n^\pm]$, is in fact a constant.*
   (b) *The quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ is given by*
$$N_{u,v}^{w,\lambda} = \sum_{\lambda_1,\lambda_2\in Q^\vee} c_{x,[t_{\lambda_1}]} c_{y,[t_{\lambda_2}]} d_{z,[t_{\lambda_1+\lambda_2}]}.$$
   Furthermore, one has the following (by simplifying the summation)
$$N_{u,v}^{w,\lambda} = \sum_{(\lambda_1,\lambda_2,v_1)\in\Gamma\times W} c_{x,[v_1 t_{\lambda_1}]} c_{y,[v_1 t_{\lambda_2}]} d_{z,[v_1 t_{\lambda_1+\lambda_2}]},$$
   where $\Gamma = \{(\lambda_1,\lambda_2) \mid \lambda_1,\lambda_2 \succcurlyeq A, \lambda_1+\lambda_2 \preccurlyeq 2A+\lambda, \lambda_1,\lambda_2 \in \tilde{Q}^\vee\}$.

In section 5.1, we review the definition of equivariant quantum cohomology of G/B. After that, we prove Theorem 5.1 in section 5.2. Finally in section 5.3, we show that our formula does recover the usual formula for equivariant Schubert structure constants for G/B.

5.1. **Equivariant quantum cohomology of $X = G/B$.** Let σ_w and σ^w denote the Schubert classes in $H_*(X,\mathbb{Z})$ and $H^*(X,\mathbb{Z})$ respectively. One has $H_*(X,\mathbb{Z}) = \bigoplus_{w\in W}\mathbb{Z}\sigma_w$; $H^*(X,\mathbb{Z}) = \bigoplus_{w\in W}\mathbb{Z}\sigma^w$; $\langle\sigma_u,\sigma^v\rangle = \delta_{u,v}$ for any $u, v\in W$. If we write $g^{u,v} = \int_{[X]}\sigma^u\cup\sigma^v$, then the matrix $\big(g^{u,v}\big)$ is invertible with its inverse denoted as $\big(g_{u,v}\big) = \big(g^{u,v}\big)^{-1}$.

For each $i \in I$, we denote $s_i = \sigma_{\alpha_i}$ and introduce a formal variable q_i. Identify $H_2(X,\mathbb{Z}) = \bigoplus_{i\in I}\mathbb{Z}\sigma_{s_i}$ with $Q^\vee$ via $\beta = \sum_i d_i\sigma_{s_i} \mapsto \lambda_\beta = \sum_i d_i\alpha_i^\vee$. Denote $q_{\lambda_\beta} = q^\beta = \prod_{i\in I} q_i^{d_i}$.

Let $\overline{\mathcal{M}_{0,m}(X,\beta)}$ be the moduli space of stable maps of degree β of m-pointed genus 0 curves into X (see [11]). Let ev_i denote the i-th canonical evaluation map $\mathrm{ev}_i : \overline{\mathcal{M}_{0,m}(X,\beta)} \to X$ given by $\mathrm{ev}_i([f : C \to X; p_1, \cdots, p_m]) = f(p_i)$. The genus zero Gromov-Witten invariant for $\gamma_1, \cdots, \gamma_m \in H^*(X) = H^*(X,\mathbb{Q})$ is defined as
$$I_{0,m,\beta}(\gamma_1,\cdots,\gamma_m) = \int_{\overline{\mathcal{M}_{0,m}(X,\beta)}} \mathrm{ev}_1^*(\gamma_1)\cup\cdots\cup\mathrm{ev}_m^*(\gamma_m).$$

The (small) quantum product for $a, b \in H^*(X)$ is a deformation of the cup product defined as follows.

$$a \star b = \sum_{u,v\in W;\beta\in H_2(X,\mathbb{Z})} I_{0,3,\beta}(a, b, \sigma^u) g_{u,v}\sigma^v q^\beta.$$

The $\mathbb{Q}[\mathbf{q}]$-module $H^*(X)[\mathbf{q}] := H^*(X)\otimes\mathbb{Q}[\mathbf{q}]$ equipped with $\star$ is called the small quantum cohomology ring of X and denoted as $QH^*(X)$. So the same Schubert classes $\sigma^u = \sigma^u \otimes 1$ form a basis for $QH^*(X)$ over $\mathbb{Q}[\mathbf{q}]$ and we write

$$\sigma^u \star \sigma^v = \sum_{w\in W,\lambda\in Q^\vee} N_{u,v}^{w,\lambda} q_\lambda \sigma^w.$$

The coefficients $N_{u,v}^{w,\lambda}$'s are called the *quantum Schubert structure constants.* In fact, $\sum_{v_1\in W} g_{v_1,w}\sigma^{v_1} = \sigma^{\omega_0 w}$ (see e.g. [12]). Compared with the original definition of quantum product, the quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ is exactly equal to the (3-pointed genus zero) Gromov-Witten invariant $I_{0,3,\lambda}(\sigma^u, \sigma^v, \sigma^{\omega_0 w})$. When $\lambda = 0$, they give the classical Schubert structure constants for $H^*(X)$. The T-action on X induces an action on the moduli space $\overline{\mathcal{M}_{0,3}(X, \beta)}$ given by: $t \cdot (f : C \to X; p_1, p_2, p_3) = (f_t : C \to X; p_1, p_2, p_3)$ where $f_t(x) := t \cdot f(x)$. The evaluation maps ev_i's are T-equivariant. We use the same notation σ^u to denote the equivariant Schubert class in $H_T^*(X)$. The equivariant Gromov-Witten invariant is defined as $I_{0,3,\beta}^T(\sigma^u, \sigma^v, \sigma^w) = \pi_*^T(\mathrm{ev}_1^T(\sigma^u) \cdot \mathrm{ev}_2^T(\sigma^v) \cdot \mathrm{ev}_3^T(\sigma^w))$, where π_*^T is the equivariant Gysin push forward. As a consequence, the equivariant (small) quantum product $\star_T$ is defined (see e.g. [33]). The equivariant quantum cohomology ring $QH_T^*(X) = (H^*(X)[\alpha, \mathbf{q}], \star_T)$ is commutative and associative, which has an $S[\mathbf{q}]$-basis of Schubert classes with $S[\mathbf{q}] = \mathbb{Q}[\alpha_1, \cdots, \alpha_n, q_1, \cdots, q_n]$. Furthermore,

$$\sigma^u \star \sigma^v = \sum_{w\in W,\lambda\in Q^\vee} \tilde{N}_{u,v}^{w,\lambda} q_\lambda \sigma^w, \quad \text{where } \tilde{N}_{u,v}^{w,\lambda} = N_{u,v}^{w,\lambda}(\alpha) \in S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n].$$

When $\lambda = 0$, $\tilde{N}_{u,v}^{w,\lambda}$ is equivalent to the corresponding equivariant Schubert structure constant. The evaluation $\tilde{N}_{u,v}^{w,\lambda}|_{\alpha_1=\cdots=\alpha_n=0}$ equals the quantum Schubert structure constant $N_{u,v}^{w,\lambda}$. A direct calculation of a general $N_{u,v}^{w,\lambda}$ can be rather difficult. However if v is a simple reflection, then the following equivariant quantum Chevalley formula holds, which was originally stated by Peterson in [35] and has been proved by Mihalcea in [33]. Furthermore, the formula completely determines the multiplication in $QH_T^*(G/B)$ as shown in [33].

Proposition 5.2 (Equivariant quantum Chevalley formula)**.** *Let $u \in W$ and $s_i = \sigma_{\alpha_i}$ with $i \in I$. Then in $QH_T^*(G/B)$ one has*

$$\sigma^{s_i} \star_T \sigma^u = (w_i - u(w_i))\sigma^u + \sum_{\gamma\in\Gamma_1} \langle\gamma^\vee, w_i\rangle \sigma^{u\sigma_\gamma} + \sum_{\gamma\in\Gamma_2} \langle\gamma^\vee, w_i\rangle q_{\gamma^\vee} \sigma^{u\sigma_\gamma},$$

where $\Gamma_1 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1\}$ and $\Gamma_2 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1 - \langle\gamma^\vee, 2\rho\rangle\}$.

By evaluating at $w_i = 0$, the quantum Chevalley formula (see [12]) is recovered.

5.2. **Explicit formula for quantum Schubert structure constants.** In this subsection, we prove Theorem 5.1 by establishing an equivalence between $QH_T^*(G/B)$ and $H_*^T(\Omega K)$ after localization. The key point is Mihalcea's criterion as follows, a special case ($P = B$) of which is stated here only.

Proposition 5.3 (Mihalcea's criterion; see Theorem 2 of [33]).
Denote $\mathbb{Q}[\alpha, \mathbf{q}] = \mathbb{Q}[\alpha_1, \cdots, \alpha_n, q_1, \cdots, q_n]$ *and* $\mathbb{Q}[\alpha^\pm, \mathbf{q}] = \mathbb{Q}[\alpha_1^\pm, \cdots, \alpha_n^\pm, q_1, \cdots, q_n]$. *Let* $\mathcal{A} = \bigoplus_{u\in W} \mathbb{Q}[\alpha^\pm, \mathbf{q}]\sigma^u$ *be any* $\mathbb{Q}[\alpha^\pm, \mathbf{q}]$*-algebra with the product written as* $\sigma^u * \sigma^v = \sum_{w,\lambda} C_{u,v}^{w,\lambda} q_\lambda \sigma^w$ *(where* $\lambda \succcurlyeq 0$*). Suppose the structure coefficients* $C_{u,v}^{w,\lambda}$*'s satisfy the following*

1. *(homogeneity)* $C_{u,v}^{w,\lambda} \in \mathbb{Q}[\alpha^\pm]$ *is a homogeneous rational function of degree*
$$\deg(C_{u,v}^{w,\lambda}) = \ell(u) + \ell(v) - \ell(w) - \langle \lambda, 2\rho\rangle, \text{ whenever } C_{u,v}^{w,\lambda} \neq 0;$$
2. *(multiplication by unit)* $C_{id,v}^{w,\lambda} = \begin{cases} 1, & \text{if } \lambda = 0 \text{ and } w = v \\ 0, & \text{otherwise}\end{cases};$
3. *(commutativity)* $\sigma^u * \sigma^v = \sigma^v * \sigma^u$ *for any* $u, v \in W$*;*
4. *(associativity)* $(\sigma^{s_i} * \sigma^u) * \sigma^v = \sigma^{s_i} * (\sigma^u * \sigma^v)$ *for any* $u, v \in W$ *and any simple reflection* $s_i \in W$*;*
5. *(equivariant quantum Chevalley formula) For any* $u \in W$ *and any simple reflection* $s_i \in W$*, the product of* $\sigma^{s_i} * \sigma^u$ *is given by*
$$\sigma^{s_i} * \sigma^u = (w_i - u(w_i))\sigma^u + \sum_{\gamma\in\Gamma_1} \langle \gamma^\vee, w_i\rangle \sigma^{u\sigma_\gamma} + \sum_{\gamma\in\Gamma_2} \langle \gamma^\vee, w_i\rangle q_{\gamma^\vee} \sigma^{u\sigma_\gamma}.$$

Then for any u, v, w, λ*, one has*
$$C_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}.$$
In particular, $C_{u,v}^{w,\lambda} = 0$ *if* $\deg(C_{u,v}^{w,\lambda}) < 0$*, and* $(\bigoplus_{u\in W} \mathbb{Q}[\alpha, \mathbf{q}]\sigma^u, *)$ *is canonically isomorphic to* $QH_T^*(G/B)$ *as* $\mathbb{Q}[\alpha, \mathbf{q}]$*-algebras.*

Remark 5.4. *As shown in* [34]*, the equivariant Schubert structure constants are in fact nonnegative combinations of monomials in the negative simple roots. Therefore, Mihalcea chose the negative simple roots instead of the positive ones for positivity reasons. For the same reason, we define the new algebra* $(H_*^T(\Omega K), \bullet)$ *below. As a consequence, the canonical isomorphism after localization between* $(H_*^T(\Omega K), \cdot)$ *and* $QH_T^*(G/B)$ *looks even more natural.*

Define a new product $\bullet$ on $H_*^T(\Omega K)$ as follows.
$$\mathfrak{S}_x \bullet \mathfrak{S}_y = \sum\nolimits_{z\in W_{\text{af}}^-} \tilde{b}_{x,y}^z \mathfrak{S}_z, \text{ where } \tilde{b}_{x,y}^z = (-1)^{\ell(z)-\ell(x)-\ell(y)} b_{x,y}^z.$$
Note that $b_{x,y}^z$ is a homogeneous polynomial of degree $\ell(z) - \ell(x) - \ell(y)$. Thus $(H_*^T(\Omega K), \bullet)$ is canonically isomorphic to $(H_*^T(\Omega K), \cdot)$ as S-algebras. Immediately, it follows from the definition of $\bullet$ and Proposition 4.1 that $\mathfrak{S}_x \bullet \mathfrak{S}_{t_\mu} = \mathfrak{S}_{xt_\mu}$ for any $x, t_\mu \in W_{\text{af}}^-$. As a consequence, $\{\mathfrak{S}_t \mid t \in \tilde{Q}^\vee\}$ is a multiplicatively closed set without zero divisors. We have the following S-module homomorphism
$$\varphi: H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee] \longrightarrow QH_T^*(G/B)[\mathbf{q}^{-1}];$$
$$\mathfrak{S}_{wt_\lambda} \bullet \mathfrak{S}_{t_\mu}^{-1} \longmapsto q_{\lambda-\mu}\sigma^w,$$
where $QH_T^*(G/B)[\mathbf{q}^{-1}] = QH_T^*(G/B)[q_i^{-1} | i \in I]$.

It is easy to show that φ is an S-module isomorphism. As a consequence, the algebra $H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee]$ has an S-basis $\{\varphi^{-1}(q_\lambda\sigma^w) \mid \lambda \in Q^\vee, w \in W\}$. Therefore for any $A, B \in H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee]$, $A \bullet B = \sum_{w,\lambda} C_{A,B}^{w,\lambda} \varphi^{-1}(q_\lambda \sigma^w)$. Furthermore, we have

Theorem 5.5. (i) *$\varphi : H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee] \longrightarrow QH_T^*(G/B)[\mathbf{q}^{-1}]$ is an S-algebra isomorphism.*

(ii) *Let $u, v, w \in W$ and $\lambda \in Q^\vee$. Take $\eta, \kappa, \mu \in \tilde{Q}^\vee$ such that $x = ut_\eta, y = vt_\kappa, z = wt_\mu$ lie in W_{af}^- and $\lambda = \mu - \eta - \kappa$. Then one has*

$$\tilde{N}_{u,v}^{w,\lambda} = \tilde{b}_{x,y}^{z}.$$

Proof. To prove (i), we note that $\varphi^{-1}(q_\lambda) \bullet \varphi^{-1}(q_\mu) = \varphi^{-1}(q_{\lambda+\mu})$ for any $\lambda, \mu \in Q^\vee$.

Let $\mathcal{A} = \bigoplus_{w\in W} \mathbb{Q}[\alpha, \varphi^{-1}(\mathbf{q})]\varphi^{-1}(\sigma^w)$, where $\mathbb{Q}[\alpha, \varphi^{-1}(\mathbf{q})] = \mathbb{Q}[\alpha_1, \cdots, \alpha_n, \varphi^{-1}(q_1), \cdots, \varphi^{-1}(q_n)]$. Clearly, $H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee] = \mathcal{A}[\varphi^{-1}(q_i^{-1}) \mid i \in I]$.

We claim that $\mathcal{A}$ is a $\mathbb{Q}[\alpha, \varphi^{-1}(\mathbf{q})]$-algebra with basis $\{\varphi^{-1}(\sigma^w) \mid w \in W\}$ and satisfies assumption (1) of Proposition 5.3. Indeed, we take $\tau \in Q^\vee$ such that $\langle \tau, \alpha_i\rangle \leq -5\ell(\omega_0)$ for all $i \in I$. Note that for $u, v \in W$, $\varphi^{-1}(\sigma^u) = \mathfrak{S}_{ut_\tau}\mathfrak{S}_{t_\tau}^{-1}$, $\varphi^{-1}(\sigma^v) = \mathfrak{S}_{vt_\tau}\mathfrak{S}_{t_\tau}^{-1}$ and $\varphi^{-1}(\sigma^u) \bullet \varphi^{-1}(\sigma^v) = \sum_{wt_\mu\in W_{\mathrm{af}}^-} \tilde{b}_{ut_\tau, vt_\tau}^{wt_\mu}\mathfrak{S}_{wt_\mu}\mathfrak{S}_{t_{2\tau}}^{-1} = \sum_{w,\lambda} C_{u,v}^{w,\lambda}\varphi^{-1}(q_\lambda\sigma^w)$. Here $C_{u,v}^{w,\lambda} = \tilde{b}_{ut_\tau,vt_\tau}^{wt_\mu} = (-1)^{\ell(wt_\mu)-\ell(ut_\tau)-\ell(vt_\tau)}b_{ut_\tau,vt_\tau}^{wt_\mu}$ with $\lambda = \mu - 2\tau$ and $\deg(C_{u,v}^{w,\lambda}) = \ell(wt_\mu) - \ell(ut_\tau) - \ell(vt_\tau) = \ell(u) + \ell(v) - \ell(w) - \langle\lambda, 2\rho\rangle$ whenever $C_{u,v}^{w,\lambda} \neq 0$. Furthermore, it follows from Proposition 4.27 that $C_{u,v}^{w,\lambda} \neq 0$ only if $\ell(wt_\mu) - \ell(ut_\tau) - \ell(vt_\tau) \geq 0$ and then $(-1)^{\ell(wt_\mu)-\ell(ut_\tau)-\ell(vt_\tau)}C_{u,v}^{w,\lambda} = b_{ut_\tau,vt_\tau}^{wt_\mu} = \sum_{v_1,\lambda_1,\lambda_2} c_{ut_\tau,[v_1t_{\lambda_1}]}c_{vt_\tau,[v_1t_{\lambda_2}]}d_{wt_\mu,[v_1t_{\lambda_1+\lambda_2}]}$ with the summation running over the set $\{(v_1, \lambda_1, \lambda_2) \in W \times \tilde{Q}^\vee \times \tilde{Q}^\vee \mid \lambda_1, \lambda_2 \succcurlyeq \tau, \lambda_1 + \lambda_2 \preccurlyeq \mu\}$. Hence, $C_{u,v}^{w,\lambda} \neq 0$ only if the above set that the summation runs over is nonempty. In particular, $\lambda = \mu - 2\tau \succcurlyeq 0$. Hence, our claim does hold.

For $s_it_\lambda, ut_\mu \in W_{\mathrm{af}}^-$, it follows from the definition of $\bullet$ and Proposition 4.2 that

$$\mathfrak{S}_{s_it_\lambda}\bullet\mathfrak{S}_{ut_\mu} = (w_i - u(w_i))\mathfrak{S}_{ut_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_1}\langle\gamma^\vee, w_i\rangle\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_2}\langle\gamma^\vee, w_i\rangle\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}}.$$

Hence, $\varphi^{-1}(\sigma^{s_i}) \bullet \varphi^{-1}(\sigma^u)$ satisfies assumption (5) of Proposition 5.3.

Note that $\mathfrak{S}_{t_\tau} \bullet \mathfrak{S}_{vt_\tau} = \mathfrak{S}_{vt_{2\tau}}$. Hence, $\varphi^{-1}(\sigma^{\mathrm{id}}) \bullet \varphi^{-1}(\sigma^v) = \varphi^{-1}(\sigma^v)$, that is, assumption (2) of Proposition 5.3 holds. Assumptions (3), (4) of Proposition 5.3 hold for $\mathcal{A}$ trivially.

Therefore for any $u, v, w \in W$ and any $\lambda \in Q^\vee$, $C_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}$ follows from Proposition 5.3. In particular, $\varphi|_{\mathcal{A}} : \mathcal{A} \to QH_T^*(G/B)$ is an S-algebra isomorphism. Hence, φ is an S-algebra isomorphism.

To prove (ii), we note that $\varphi^{-1}(\sigma^u) = \mathfrak{S}_x\mathfrak{S}_{t_\eta}^{-1}$, $\varphi^{-1}(\sigma^v) = \mathfrak{S}_y\mathfrak{S}_{t_\kappa}^{-1} \in \mathcal{A}$, and $\varphi^{-1}(\sigma^u) \bullet \varphi^{-1}(\sigma^v) = \sum_{wt_\mu\in W_{\mathrm{af}}^-} \tilde{b}_{x,y}^{wt_\mu}\mathfrak{S}_{wt_\mu}\mathfrak{S}_{t_{\eta+\kappa}}^{-1} = \sum_{w,\lambda} C_{u,v}^{w,\lambda}\varphi^{-1}(q_\lambda\sigma^w)$, where $\lambda = \mu - \eta - \kappa$. From (i), we have $\tilde{b}_{x,y}^{wt_\mu} = C_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}$. □

Using the above theorem, we can prove Theorem 5.1 as follows.

Proof of Theorem 5.1. Note that $N_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}\big|_{\alpha_1=\cdots=\alpha_n=0}$, the evaluation of the equivariant quantum Schubert structure constant $\tilde{N}_{u,v}^{w,\lambda} \in \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ at the origin $(\alpha_1, \cdots \alpha_n) = (0, \cdots, 0)$.

It follows from Theorem 5.5 that $\tilde{N}_{u,v}^{w,\lambda} = (-1)^{\ell(z)-\ell(x)-\ell(y)}b_{x,y}^z$ for $x, y, z \in W_{\mathrm{af}}^-$ with $x = ut_\eta, y = vt_\kappa, z = wt_\mu$ and $\lambda = \mu - \eta - \kappa$. Note that $b_{x,y}^z$ is a homogeneous (rational) polynomial of degree $\ell(z) - \ell(x) - \ell(y) = \ell(t_\mu) - \ell(w) - (\ell(t_\eta) - \ell(u)) - (\ell(t_\kappa) - \ell(v)) = \ell(u) + \ell(v) - \ell(w) - \langle\lambda, 2\rho\rangle$.

(1) If $\langle\lambda, 2\rho\rangle > \ell(u)+\ell(v)-\ell(w)$, then it follows form Corollary 3.9 that $b^z_{x,y}=0$ and therefore $N^{w,\lambda}_{u,v}=0$. If $\langle\lambda, 2\rho\rangle < \ell(u)+\ell(v)-\ell(w)$, then $b^z_{x,y}$ is a homogeneous polynomial of positive degree $\ell(z)-\ell(x)-\ell(y)>0$. The evaluation of $b^z_{x,y}$ at the origin is 0, and therefore $N^{w,\lambda}_{u,v}=0$.

(2) If $\langle\lambda, 2\rho\rangle = \ell(u)+\ell(v)-\ell(w)$, then $b^z_{x,y}$ is a constant polynomial. In particular,

$$N^{w,\lambda}_{u,v}=\tilde N^{w,\lambda}_{u,v}\big|_{\alpha_1=\cdots=\alpha_n=0}=b^z_{x,y}\big|_{\alpha_1=\cdots=\alpha_n=0}=b^z_{x,y}\big|_{\alpha_1=\cdots=\alpha_n=1}.$$

Take $\eta,\kappa,\mu\in\tilde Q^\vee$ such that $x,y,z\in W^-_{\rm af}$ with $x=ut_\eta, y=vt_\kappa, z=wt_\mu$ and $\lambda=\mu-\eta-\kappa$. This can be done as follows.

The possible determinant of the Cartan matrix $\big(\langle\alpha_i^\vee,\alpha_j\rangle\big)$ is $1,2,3,4$ and $n+1$ (see e.g. section 13 of [15]). As a consequence, the element $A=-12n(n+1)\sum_{i\in I}w_i^\vee$ is in the coroot lattice $Q^\vee$. Furthermore, $A\in\tilde Q^\vee$ and $\langle A,\alpha_i\rangle=-12n(n+1)<-5|R^+|=-5\ell(\omega_0)$ (see e.g. [15]). Note that $\lambda=\sum_{i\in I}a_i\alpha_i^\vee\succcurlyeq 0$ and $\langle\lambda,\alpha_i\rangle\leq 2a_i\leq\langle\lambda,2\rho\rangle=\ell(u)+\ell(v)-\ell(w)\leq 2\ell(\omega_0)$. Thus $\langle 2A+\lambda,\alpha_i\rangle<0$ for each $i\in I$. Let $x=ut_A, y=vt_A$ and $z=wt_{2A+\lambda}$. Then $x,y,z\in W^-_{\rm af}$ and $\lambda=(2A+\lambda)-A-A$.

Hence, the first formula follows from Theorem 3.3 and Theorem 5.5, and the second formula follows from Proposition 4.27 immediately.

□

Using Theorem 5.5 again, we obtain the following

Proposition 5.6. *For any $u,v,w\in W$ and $\lambda\in Q^\vee$ with $\lambda\succcurlyeq 0$, we take $\eta,\kappa\in\tilde Q^\vee$ such that $x=ut_\eta$ and $y=vt_\kappa$ lie in $W^-_{\rm af}$ and we denote $\mu=\eta+\kappa+\lambda$. If $wt_\mu\notin W^-_{\rm af}$, then the equivariant quantum Schubert structure constant $\tilde N^{w,\lambda}_{u,v}$ vanishes.*

Proof. Denote $d=\ell(u)+\ell(v)-\ell(w)-\langle\lambda,2\rho\rangle$. Take $M\in\mathbb N$ with $12(n+1)|M$ and $M\gg 0$ such that $B=-M\sum_{i\in I}w_i^\vee\in\tilde Q^\vee$, which does exist (following the proof of Theorem 5.1), and $\mu+2B\in\tilde Q^\vee$ is regular. Then $xt_B, yt_B, wt_{\mu+2B}\in W^-_{\rm af}$. Therefore it follows from Theorem 5.5 that $\tilde N^{w,\lambda}_{u,v}=(-1)^d b^{wt_{\mu+2B}}_{xt_B,yt_B}$. On the other hand, it follows from Proposition 4.1 that

$$\mathfrak S_{xt_B}\mathfrak S_{yt_B}=\mathfrak S_{t_{2B}}\mathfrak S_x\mathfrak S_y=\mathfrak S_{t_{2B}}\sum\nolimits_{z\in W^-_{\rm af}}b^z_{x,y}\mathfrak S_z=\sum\nolimits_{z\in W^-_{\rm af}}b^z_{x,y}\mathfrak S_{zt_{2B}}.$$

Therefore for $zt_{2B}\in W^-_{\rm af}$, $b^{zt_{2B}}_{xt_B,yt_B}\neq 0$ only if $z\in W^-_{\rm af}$. Hence, $\tilde N^{w,\lambda}_{u,v}=0$. □

As we will see in section 6.1, Proposition 5.6 is useful when we need to compute the quantum Schubert structure constants for G/B by hand when the rank of G is not too big.

5.3. **Equivariant Schubert structure constants.** In this subsection, we show that the equivariant quantum Schubert structure constants $\tilde N^{w,0}_{u,v}$'s indeed give the equivariant Schubert structure constants for G/B.

Let $u,v,w\in W$. Take $\lambda=\mu=-12n(n+1)\sum_{i\in I}w_i^\vee\in\tilde Q^\vee$. Denote $x=ut_\lambda, y=vt_\mu$ and $z=wt_{\lambda+\mu}$. Clearly, $x,y,z\in W^-_{\rm af}$. It follows from Theorem 5.5 that $\tilde N^{w,0}_{u,v}=(-1)^d b^z_{x,y}$, where $d=\ell(z)-\ell(x)-\ell(y)=\ell(u)+\ell(v)-\ell(w)\geq 0$. By Proposition 4.27, $b^z_{x,y}=\sum_{v_1\in W;\lambda_1,\lambda_2}c_{x,[v_1t_{\lambda_1}]}c_{y,[v_1t_{\lambda_2}]}d_{z,[v_1t_{\lambda_1+\lambda_2}]}$, where (λ_1,λ_2)

runs over the set $\{(\lambda_1,\lambda_2)\in\tilde{Q}^\vee\times\tilde{Q}^\vee \mid \lambda_1\succcurlyeq\lambda, \lambda_2\succcurlyeq\mu, \lambda_1+\lambda_2\preccurlyeq(\lambda+\mu)\}=\{(\lambda,\mu)\}$. Hence, it follows from Lemma 4.16 that

$$b^z_{x,y}=\sum_{v_1\in W}c_{ut_\lambda,[v_1t_\lambda]}c_{vt_\mu,[v_1t_\mu]}d_{wt_{\lambda+\mu},[v_1t_{\lambda+\mu}]}=\sum_{v_1\in W}c'_{ut_\lambda,v_1t_\lambda}c'_{vt_\mu,v_1t_\mu}d_{wt_{\lambda+\mu},[v_1t_{\lambda+\mu}]}.$$

Note that $\langle\lambda,\alpha_i\rangle\le -2$ and $\langle\mu,\alpha_i\rangle\le -2$ for all $i\in I$. Use the same notations of H_j's and $\tilde{H}_k$'s as in section 4.4. By Lemma 4.23, we have

$$\begin{aligned}
&c'_{ut_\lambda,v_1t_\lambda}c'_{vt_\mu,v_1t_\mu}d_{wt_{\lambda+\mu},[v_1t_{\lambda+\mu}]}\\
=&\frac{v_1(d_{u^{-1},v_1^{-1}})}{\prod_{j=1}^m v_1(H_j)}\cdot\frac{v_1(d_{v^{-1},v_1^{-1}})}{\prod_{j=1}^p v_1(\tilde{H}_j)}\cdot d_{w\omega_0,v_1\omega_0}\prod_{j=r+1}^{m+p}v_1(H_j)\big|_{\alpha_0=-\theta}\\
=&\frac{v_1(d_{u^{-1},v_1^{-1}})\cdot v_1(d_{v^{-1},v_1^{-1}})}{v_1\Big(\prod_{\gamma\in R^+}\gamma\Big)}\cdot d_{w\omega_0,v_1\omega_0}\\
=&\frac{(-1)^{\ell(u)}d_{u,v_1}\cdot(-1)^{\ell(v)}d_{v,v_1}}{(-1)^{\ell(v_1)}\prod_{\gamma\in R^+}\gamma}\cdot(-1)^{\ell(w\omega_0)}v_1\omega_0 d_{(w\omega_0)^{-1},(v_1\omega_0)^{-1}} && (*)\\
=&\frac{(-1)^{\ell(u)+\ell(v)+\ell(\omega_0)-\ell(w)}d_{u,v_1}d_{v,v_1}}{(-1)^{\ell(v_1)}\prod_{\gamma\in R^+}\gamma}\cdot v_1\omega_0\Big(\big(\prod_{\beta\in R^+}\beta\big)(v_1\omega_0)^{-1}(c_{(w\omega_0)\cdot\omega_0,(v_1\omega_0)\cdot\omega_0}\Big) && (**)\\
=&(-1)^{\ell(u)+\ell(v)+\ell(\omega_0)-\ell(w)-\ell(v_1)}d_{u,v_1}d_{v,v_1}\cdot(-1)^{\ell(v_1\omega_0)}c_{w,v_1}\\
=&(-1)^{\ell(u)+\ell(v)-\ell(w)}d_{u,v_1}d_{v,v_1}c_{w,v_1}.
\end{aligned}$$

Hence,

$$\tilde{N}^{w,0}_{u,v}=(-1)^{\ell(u)+\ell(v)-\ell(w)}b^z_{x,y}=\sum\nolimits_{v_1\in W}d_{u,v_1}d_{v,v_1}c_{w,v_1}.$$

This formula does agree with the formula for the equivariant Schubert structure constants for G/B as in Remark 7.5.

Note that $(**)$ follows from Lemma 4.21 and $(*)$ follows from the general fact that $d_{v,u}=(-1)^{\ell(v)}u(d_{v^{-1},u^{-1}})$ for any $v,u\in W$. Indeed, let $u=[s_1\cdots s_k]_{\rm red}$ where $s_j=\sigma_{\beta_j}$ with $\beta_j\in\Delta$. Note that $u^{-1}=[s_k\cdots s_1]_{\rm red}$. Denote $a=\ell(v)$ and note that $v=[s_{i_1}\cdots s_{i_a}]_{\rm red}\iff v^{-1}=[s_{i_a}\cdots s_{i_1}]_{\rm red}$. Therefore by definition,

$$\begin{aligned}
u(d_{v^{-1},u^{-1}})&=u\Big(\prod_{v^{-1}=[s_{i_a}\cdots s_{i_1}]_{\rm red}}\prod_{j=1}^a s_{i_a}\cdots s_{i_j+1}(\beta_{i_j})\Big)\\
&=\prod_{v=[s_{i_1}\cdots s_{i_a}]_{\rm red}}\prod_{j=1}^a -s_{i_1}\cdots s_{i_j-1}(\beta_{i_j})=(-1)^{\ell(v)}d_{v,u}.
\end{aligned}$$

6. Examples

In this section, we give two examples to demonstrate the effectiveness of our formula. To make the procedure precise, the first example is simple and includes some more explanations.

6.1. **Type A_2.** $G = SL(3,\mathbb{C})$; $B \subset G$ consists of upper triangular matrices in G. In this case, $X = G/B = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim_{\mathbb{C}} V_i = i, i = 1, 2\}$.

$\Delta = \{\alpha_1, \alpha_2\}, R^+ = \{\alpha_1, \alpha_2, \theta = \alpha_1 + \alpha_2\}$. Denote $s_i = \sigma_{\alpha_i}$, then one has $W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \cong S_3$. $\sigma^{s_1s_2} \star \sigma^{s_1s_2}$, $\sigma^{s_1s_2} \star \sigma^{s_2s_1}$, $\sigma^{s_2s_1} \star \sigma^{s_2s_1}$, $\sigma^{s_1s_2} \star \sigma^{s_1s_2s_1}$, $\sigma^{s_2s_1} \star \sigma^{s_1s_2s_1}$ and $\sigma^{s_1s_2s_1} \star \sigma^{s_1s_2s_1} \in QH^*(X)$ are the only products that are not given by the quantum Chevalley formula directly. As an application of our theorems, we compute one of them in details as follows.

***General discussion**: For $u, v \in W$ with $\ell(u) \geq 2$ and $\ell(v) \geq 2$, $\sigma^u \star \sigma^v = \sum_{w,\lambda} N_{u,v}^{w,\lambda} q_\lambda \sigma^w$. Note that $N_{u,v}^{w,\lambda} = 0$ unless $\lambda = a_1\alpha_1^\vee + a_2\alpha_2^\vee \succcurlyeq 0$ and $2(a_1+a_2) = \langle \lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w) \geq 4 - \ell(w) \geq 1$, in which case $q_\lambda = q_1^{a_1} q_2^{a_2}$. Note that $-\theta^\vee = -\alpha_1^\vee - \alpha_2^\vee \in \tilde{Q}^\vee$ is regular. Therefore, $x = ut_{-\theta^\vee}, y = vt_{-\theta^\vee} \in W_{\mathrm{af}}^-$. Let $z = wt_{-2\theta^\vee + \lambda}$. By Proposition 5.6, Theorem 5.5 and Theorem 5.1, we have the following*

(i) *If $z \notin W_{\mathrm{af}}^-$ or $\ell(z) \neq \ell(x) + \ell(y)$, then $N_{u,v}^{w,\lambda} = 0$.*
(ii) *If $z \in W_{\mathrm{af}}^-$, then*

$$N_{u,v}^{w,\lambda} = b_{x,y}^z = \sum_{t_1, t_2 \in Q^\vee} c_{x,[t_1]} c_{y,[t_2]} d_{z,[t_1t_2]} = \sum c_{x,[v_1t_{\lambda_1}]} c_{y,[v_2t_{\lambda_2}]} d_{z,[v_2 t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}]},$$

where the effective summation runs over those $v_it_{\lambda_i} = m_{[t_i]} \in W_{\mathrm{af}}^-$ satisfying $v_1t_{\lambda_1} \preccurlyeq x$ and $v_2t_{\lambda_2} \preccurlyeq y$. Furthermore if $x \neq 1$, then $d_{z,[t_2]} = 0$ as $\ell(z) > \ell(y) \geq \ell([t_2])$. In particular if $x, y \neq 1$, then we do not need to consider the case $v_it_{\lambda_i} = 1$.

Calculation for the case $u = s_1s_2$ and $v = s_1s_2s_1$.

In this case, $x = s_1s_2t_{-\theta^\vee} = s_2s_0$ and $y = s_1s_2s_1t_{-\theta^\vee} = s_0$. $\lambda = a_1\alpha_1^\vee + a_2\alpha_2^\vee$ with $a_1, a_2 \geq 0$ and $2(a_1 + a_2) = \ell(u) + \ell(v) - \ell(w) = 5 - \ell(w)$. Hence, $(a_1, a_2) = (1,1), (2,0), (0,2), (1,0)$ or $(0,1)$.

If $(a_1, a_2) = (2, 0)$, then $z \notin W_{\mathrm{af}}^-$ by noting $-2\theta^\vee + \lambda = 2\alpha_2^\vee \notin \tilde{Q}^\vee$. If $(a_1, a_2) = (1, 0)$, then $\lambda = \alpha_1$, $\ell(w) = 3$ and $w = s_1s_2s_1$. Since $\langle -2\theta^\vee + \alpha_1, \alpha_1\rangle = 0$ while $w(\alpha_1) = -\alpha_2 \notin R^+$, $z \notin W_{\mathrm{af}}^-$. Similarly, we can show $z \notin W_{\mathrm{af}}^-$ if $(a_1, a_2) = (0, 2)$ or $(0, 1)$. Hence, $N_{u,v}^{w,\lambda} = 0$ unless $(a_1, a_2) = (1, 1)$.

If $(a_1, a_2) = (1, 1)$, then $\ell(w) = 5 - 2(1 + 1) = 1$ and therefore $w = s_1$ or s_2. Hence, $\sigma^u \star \sigma^v = C_1q_1q_2\sigma^{s_1} + C_2q_1q_2\sigma^{s_2}$ for some real numbers C_1 and C_2.

Note that $1 \neq v_1t_{\lambda_1} \preccurlyeq x = s_2s_0$ with $v_1t_{\lambda_1} \in W_{\mathrm{af}}^-$ implies that $v_1t_{\lambda_1} = s_0$ or s_2s_0. $1 \neq v_2t_{\lambda_2} \preccurlyeq y = s_0$ with $v_2t_{\lambda_2} \in W_{\mathrm{af}}^-$ implies that $v_2t_{\lambda_2} = s_0 = y$. Hence,

$$\begin{aligned} b_{x,y}^z &= c_{x,[s_0]} c_{y,[y]} d_{z,[\sigma_\theta t_{-\theta^\vee - \theta^\vee}]} + c_{x,[s_2s_0]} c_{y,[y]} d_{z,[\sigma_\theta t_{\sigma_\theta s_1 s_2(-\theta^\vee) - \theta^\vee}]} \\ &= c'_{s_2s_0,s_0} c'_{s_0,s_0} d_{z,[\sigma_\theta t_{-2\theta^\vee}]} + c'_{s_2s_0,s_2s_0} c'_{s_0,s_0} d_{z,[\sigma_\theta t_{-\alpha_2^\vee - \theta^\vee}]}. \end{aligned}$$

Note that
$$c'_{s_0,s_0} = (-1)^1 \cdot \frac{1}{s_0(\alpha_0)}\Big|_{\alpha_0 = -\theta} = -\frac{1}{\theta};$$
$$c'_{s_2s_0,s_0} = (-1)^2 \cdot \frac{1}{\alpha_2 s_0(\alpha_0)}\Big|_{\alpha_0 = -\theta} = \frac{1}{\alpha_2\theta};$$
$$c'_{s_2s_0,s_2s_0} = (-1)^2 \frac{1}{s_2(\alpha_2)s_2s_0(\alpha_0)}\Big|_{\alpha_0 = -\theta} = \frac{1}{\alpha_2 s_2(\alpha_0)}\Big|_{\alpha_0 = -\theta} = -\frac{1}{\alpha_2\alpha_1}.$$

Note that $\sigma_\theta t_{-2\theta^\vee} = [s_0s_2s_1s_2s_0]_{\mathrm{red}}$ and $\sigma_\theta t_{-\alpha_2^\vee - \theta^\vee} = [s_0s_2s_1s_0s_1]_{\mathrm{red}}$.

Now for $w = s_1$ and $\lambda = \theta^\vee$, we have $z = s_1 t_{-\theta^\vee} = [s_2 s_1 s_0]_{\rm red}$. Therefore

$$d_{z,[\sigma_\theta t_{-2\theta^\vee}]} = d_{s_2s_1s_0,[s_0s_2s_1s_2s_0]} = s_0(\alpha_2)s_0s_2(\alpha_1)s_0s_2s_1s_2(\alpha_0)\big|_{\alpha_0=-\theta} = -\alpha_1\theta^2;$$
$$d_{z,[\sigma_\theta t_{-\alpha_2^\vee-\theta^\vee}]} = d_{s_2s_1s_0,[s_0s_2s_1s_0s_1]} = s_0(\alpha_2)s_0s_2(\alpha_1)s_0s_2s_1(\alpha_0)\big|_{\alpha_0=-\theta} = -\alpha_1^2\theta.$$

Hence, $\quad C_1 = b^z_{x,y} = \dfrac{1}{\alpha_2\theta}\cdot\dfrac{-1}{\theta}\cdot(-\alpha_1\theta^2)+\dfrac{-1}{\alpha_2\alpha_1}\dfrac{-1}{\theta}\cdot(-\alpha_1^2\theta) = \dfrac{\alpha_1}{\alpha_2}+(-\dfrac{\alpha_1}{\alpha_2}) = 0.$

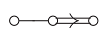
Now for $w = s_2$ and $\lambda = \theta^\vee$, we have $z = s_2 t_{-\theta^\vee} = [s_1s_2s_0]_{\rm red}$. Note that $d_{z,[\sigma_\theta t_{-2\theta^\vee}]} = d_{s_1s_2s_0,[s_0s_2s_1s_2s_0]} = s_0s_2(\alpha_1)s_0s_2s_1(\alpha_2)s_0s_2s_1s_2(\alpha_0)\big|_{\alpha_0=-\theta} = -\alpha_2\theta^2$ and that $d_{z,[\sigma_\theta t_{-\alpha_2^\vee-\theta^\vee}]} = d_{s_1s_2s_0,[s_0s_2s_1s_0s_1]} = 0$ as $s_1s_2s_0 \not\preccurlyeq s_0s_2s_1s_0s_1$. Therefore, $C_2 = b^z_{x,y} = \frac{1}{\alpha_2\theta}\cdot\frac{-1}{\theta}\cdot(-\alpha_2\theta^2) = 1$.

Hence,

$$\sigma^{s_1s_2}\star\sigma^{s_1s_2s_1} = q_1q_2\sigma^{s_2}.$$

Similarly, we can compute quantum products for the remaining cases.

6.2. **Type** B_3**.** $G = Spin(7,\mathbb{C})$; $X = G/B = \{V_1 \leqslant V_2 \leqslant V_3 \leqslant \mathbb{C}^7 \mid \dim_{\mathbb{C}} V_i = i, (V_i, V_i) = 0, i = 1,2,3\}$, where $(\cdot,\cdot)$ is a quadratic form on $\mathbb{C}^7$. See e.g. [15] for



; $\quad \theta = \alpha_1+2\alpha_2+2\alpha_3 = s_2s_3s_2(\alpha_1), \quad \theta^\vee = \alpha_1^\vee+2\alpha_2^\vee+\alpha_3^\vee, \quad |R^+| = 9,$

W is generated by simple reflections $\{s_1, s_2, s_3\}$, $|W| = 48$, $\sigma_\theta = [s_2s_3s_2s_1s_2s_3s_2]_{\rm red}$.

Calculation for $\sigma^u\star\sigma^v$, where $u = s_1s_2s_3s_1s_2$ and $v = s_3s_1s_2s_3s_1s_2$.

Note that $\langle-\theta^\vee, \alpha_2\rangle = -1 < 0$, $\langle-\theta^\vee,\alpha_i\rangle = 0$ while $u(\alpha_i) \succcurlyeq 0$ and $v(\alpha_i) \succcurlyeq 0$ for $i = 1, 3$. Hence, it suffices to take $A = -\theta^\vee$. Then one has $x = ut_{-\theta^\vee} = [s_3s_2s_0]_{\rm red}$ and $y = vt_{-\theta^\vee} = [s_2s_0]_{\rm red}$. Note that $u(-\theta^\vee) = \alpha_1^\vee+\alpha_2^\vee$ and $v(-\theta^\vee) = \alpha_1^\vee+\alpha_2^\vee+\alpha_3^\vee$. Denote m_t the length-minimizing element in the coset tW for $t \in Q^\vee$, then one has

$$b^z_{x,y} = \sum c_{x,[v_1t_{\lambda_1}]}c_{y,[v_2t_{\lambda_2}]}d_{z,[m_{t_1t_2}]},$$

where $v_it_{\lambda_i} = m_{t_i} \in W_{\rm af}^-$ and the effective summation runs over those satisfying $1 \neq v_1t_{\lambda_1} \preccurlyeq x$ and $1 \neq v_2t_{\lambda_2} \preccurlyeq y$. Explicitly, the possible nonzero terms are listed in the following table, where we denote $\eta = -2\alpha_1^\vee-3\alpha_2^\vee-2\alpha_3^\vee$, $\kappa = -2\alpha_1^\vee-2\alpha_2^\vee-\alpha_3^\vee$.

$([v_1t_{\lambda_1}],[v_2t_{\lambda_2}])$	$t_1\cdot t_2$	$m_{t_1t_2}$	$[m_{t_1t_2}]_{\rm red}$
$([s_0],[s_0])$	$t_{\theta^\vee+\theta^\vee}$	$\sigma_\theta t_{-2\theta^\vee}$	$s_0s_2s_3s_2s_1s_2s_3s_2s_0$
$([s_0],[s_2s_0])$	$t_{\theta^\vee+v(-\theta^\vee)}$	$s_1s_2s_3s_2s_1s_3s_2s_3t_\eta$	$s_0s_2s_3s_1s_2s_0$
$([s_2s_0],[s_0])$	$t_{v(-\theta^\vee)+\theta^\vee}$	$s_1s_2s_3s_2s_1s_3s_2s_3t_\eta$	$s_0s_2s_3s_1s_2s_0$
$([s_2s_0],[s_2s_0])$	$t_{v(-\theta^\vee)+v(-\theta^\vee)}$	$vt_{-2\theta^\vee}$	$s_2s_0s_2s_3s_2s_1s_2s_3s_2s_0$
$([s_2s_1s_0],[s_0])$	$t_{u(-\theta^\vee)+\theta^\vee}$	$s_1s_2s_1s_3s_2s_1s_3t_\eta$	$s_0s_3s_2s_3s_1s_2s_0$
$([s_2s_1s_0],[s_2s_0])$	$t_{u(-\theta^\vee)+v(-\theta^\vee)}$	$s_1s_2s_3s_2s_1t_\kappa$	$s_0s_2s_3s_2s_0$

Note that $s_3s_0 = s_0s_3 \in s_0W$. By definition,

$$\begin{aligned}
c_{x,[s_0]} &= c'_{s_3s_2s_0,s_0} + c'_{s_3s_2s_0,s_3s_0} = \Big(\frac{(-1)^3}{\alpha_3\alpha_2 s_0(\alpha_0)} + \frac{(-1)^3}{s_3(\alpha_3)s_3(\alpha_2)s_3s_0(\alpha_0)}\Big)\Big|_{\alpha_0=-\theta}\\
&= \frac{-2}{\alpha_2\theta(\alpha_2+2\alpha_3)};\\
c_{x,[s_2s_0]} &= c'_{s_3s_2s_0,s_2s_0} = \frac{(-1)^3}{\alpha_3 s_2(\alpha_2)s_2s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{1}{\alpha_2\alpha_3(\alpha_1+\alpha_2+2\alpha_3)};\\
c_{x,[s_3s_2s_0]} &= c'_{s_3s_2s_0,s_3s_2s_0} = \frac{(-1)^3}{s_3(\alpha_3)s_3s_2(\alpha_2)s_3s_2s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{-1}{\alpha_3(\alpha_2+2\alpha_3)(\alpha_1+\alpha_2)};\\
c_{y,[s_0]} &= c'_{s_2s_0,s_0} = \frac{(-1)^2}{\alpha_2 s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{1}{\alpha_2\theta};\\
c_{y,[s_2s_0]} &= c'_{s_2s_0,s_2s_0} = \frac{(-1)^2}{s_2(\alpha_2)s_2s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{-1}{\alpha_2(\alpha_1+\alpha_2+2\alpha_3)}.
\end{aligned}$$

Let $z = wt_{-2\theta^\vee+\lambda}$. Then $N_{u,v}^{w,\lambda} \neq 0$ only if $z \in W_{\rm af}^-$ and $\ell(z) = \ell(x) + \ell(y) = 5$. Note that the only possibilities are $z = s_2s_3s_1s_2s_0$, $s_1s_2s_3s_2s_0$ or $s_0s_2s_3s_2s_0$.

For $z = s_1s_2s_3s_2s_0 = wt_{-2\theta^\vee+\lambda}$, we have $w = s_2s_3s_2$ and $\lambda = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$. Note that $d_{z,[t_1t_2]} \neq 0$ only if $z \preccurlyeq m_{t_1t_2}$. Thus only $d_{z,[\sigma_\theta t_{-2\theta^\vee}]}$ and $d_{z,[vt_{-2\theta^\vee}]}$ are nonzero. Furthermore, we have

$$d_{z,[\sigma_\theta t_{-2\theta^\vee}]} = -\alpha_2\theta^2(\alpha_2+\alpha_3)(\alpha_2+2\alpha_3), \quad d_{z,[vt_{-2\theta^\vee}]} = \alpha_2\alpha_3(\alpha_2+2\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2.$$

Hence,

$$\begin{aligned}
b_{x,y}^z &= c_{x,[s_0]}c_{y,[s_0]}d_{z,[\sigma_\theta t_{-2\theta^\vee}]} + c_{x,[s_2s_0]}c_{y,[s_2s_0]}d_{z,[vt_{-2\theta^\vee}]}\\
&= \frac{2\alpha_2\theta^2(\alpha_2+\alpha_3)(\alpha_2+2\alpha_3)}{\alpha_2\theta(\alpha_2+2\alpha_3)\cdot\alpha_2\theta} - \frac{\alpha_2\alpha_3(\alpha_2+2\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2}{\alpha_2\alpha_3(\alpha_1+\alpha_2+2\alpha_3)\cdot\alpha_2(\alpha_1+\alpha_2+2\alpha_3)}\\
&= \frac{2(\alpha_2+\alpha_3)}{\alpha_2} - \frac{\alpha_2+2\alpha_3}{\alpha_2} = 1.
\end{aligned}$$

For $z = s_2s_3s_1s_2s_0 = wt_{-2\theta^\vee+\lambda}$, we have $w = s_3s_1s_2$ and $\lambda = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$.

$$\begin{aligned}
d_{z,[t_{2\theta^\vee}]} &= -2\theta^3(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3);\\
d_{z,[t_{\theta^\vee+v(-\theta^\vee)}]} &= -\theta(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2;\\
d_{z,[t_{v(-2\theta^\vee)}]} &= -\alpha_3(\alpha_1+\alpha_2+2\alpha_3)^2(2\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2+6\alpha_1\alpha_3+4\alpha_2\alpha_3+4\alpha_3^2);\\
d_{z,[t_{u(-\theta^\vee)+\theta^\vee}]} &= -\alpha_2\theta(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+\alpha_3);\\
d_{z,[t_{u(-\theta^\vee)+v(-\theta^\vee)}]} &= 0.
\end{aligned}$$

Substituting them in the summation for $b_{x,y}^z$ and simplifying, we obtain $b_{x,y}^z = 1$.

For $z = s_0s_2s_3s_2s_0 = wt_{-2\theta^\vee+\lambda}$, we have $w = s_1s_2s_3s_2s_1$ and $\lambda = 2\alpha_2^\vee + \alpha_3^\vee$.

$$\begin{aligned}
d_{z,[t_{2\theta^\vee}]} &= -\theta^3(\alpha_1^2+3\alpha_1\alpha_2+3\alpha_2^2+3\alpha_1\alpha_3+6\alpha_2\alpha_3+2\alpha_3^2);\\
d_{z,[t_{\theta^\vee+v(-\theta^\vee)}]} &= -\theta^2(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2;\\
d_{z,[t_{v(-2\theta^\vee)}]} &= -(\alpha_1+\alpha_2+2\alpha_3)^2(\alpha_1^2+2\alpha_1\alpha_2+3\alpha_1\alpha_3+\alpha_2\alpha_3+2\alpha_3^2);\\
d_{z,[t_{u(-\theta^\vee)+\theta^\vee}]} &= -\theta^2(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+\alpha_3);\\
d_{z,[t_{u(-\theta^\vee)+v(-\theta^\vee)}]} &= -\alpha_1\theta(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3).
\end{aligned}$$

Substituting them in the summation for $b_{x,y}^z$ and simplifying, we obtain $b_{x,y}^z = 1$.

Hence, we obtain the following

$$\sigma^{s_1s_2s_3s_1s_2}\star\sigma^{s_3s_1s_2s_3s_1s_2}=q_1q_2^2q_3(\sigma^{s_2s_3s_2}+\sigma^{s_3s_1s_2})+q_2^2q_3\sigma^{s_1s_2s_3s_2s_1}.$$

7. Appendix

7.1. **Proof of Proposition 4.15.** Note that $z\preccurlyeq[t_1t_2]$ implies $\ell(z)\leq\ell([t_1t_2])$. By definition, $v_2t_{\lambda_2}=m_{[t_2]}\preccurlyeq t_\mu$. It follows from Corollary 4.9 that $\mu\preccurlyeq\lambda_2$. Therefore, $\lambda_2=\mu+\kappa$ with $\kappa=\sum_{i\in I}c_i\alpha_i^\vee$, in which $c_i\geq 0$ for each $i\in I=\{1,\cdots,n\}$. If $\sum_{i\in I}2c_i>\ell(\omega_0)$, then $\ell(t_{\lambda_2})=\langle\mu+\kappa,-2\rho\rangle=\ell(t_\mu)-\sum_{i\in I}2c_i<\ell(t_\mu)-\ell(\omega_0)$, which deduces a contradiction as follows.

$$\begin{aligned}\ell(x)+\ell(y)\leq\ell([t_1t_2])&=\ell([v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}])\\&\leq\ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\&\leq\ell(v_1t_{\lambda_1})+\ell(v_1^{-1}v_2)+\ell(t_{\lambda_2})\\&\leq\ell(x)+\ell(\omega_0)+\ell(t_{\lambda_2})\\&<\ell(x)+\ell(t_\mu)=\ell(x)+\ell(y).\end{aligned}$$

If $\sum_{i\in I}2c_i\leq\ell(\omega_0)$, then for each $j\in I$ one has $\langle\lambda_2,\alpha_j\rangle=\langle\mu+\sum_{i\in I}c_i\alpha_i^\vee,\alpha_j\rangle<-\ell(\omega_0)+2c_j\leq 0$. Hence, $\lambda_2\in\tilde{Q}^\vee$ is regular. Therefore $v_1^{-1}v_2t_{\lambda_2}\in W_{\mathrm{af}}^-$ and

$$\begin{aligned}\ell(x)+\ell(y)\leq\ell(z)\leq\ell([t_1t_2])&=\ell([v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}])\\&\leq\ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\&\leq\ell(v_1t_{\lambda_1})+\ell(v_1^{-1}v_2t_{\lambda_2})\\&\leq\ell(x)+\ell(t_{\lambda_2})-\ell(v_1^{-1}v_2)\\&\leq\ell(x)+\ell(t_\mu)-0=\ell(x)+\ell(y).\end{aligned}$$

Hence, all inequalities are indeed equalities. Thus we have

$$\ell(z)=\ell([t_1t_2]);\quad\ell(v_1^{-1}v_2)=0;\quad v_1t_{\lambda_1}=x=wt_\lambda;\quad\lambda_2=\mu.$$

Therefore,

$$v_1=v_2=w;\quad\lambda_1=\lambda,\ \lambda_2=\mu;\quad z=wt_{\lambda+\mu}.$$

7.2. **Proof of Proposition 4.19.** Since **Case A** holds, it follows from the proof of Lemma 4.18 that $\lambda_1=\lambda$ and $v_2t_{\lambda_2}=ut_\mu$. Furthermore,

$$\begin{aligned}\ell(x)+\ell(y)+1=\ell([t_1t_2])=\ell([t_2t_1])&=\ell([ut_\mu u^{-1}v_1t_\lambda])\\&\leq\ell(ut_\mu u^{-1}v_1t_\lambda)\\&\leq\ell(ut_\mu)+\ell(u^{-1}v_1t_\lambda)\\&=\ell(y)+\ell(t_\lambda)-\ell(u^{-1}v_1)\\&=\ell(y)+\ell(x)+1-\ell(u^{-1}v_1).\end{aligned}$$

Hence, $\ell(u^{-1}v_1)=0$ and therefore $v_1=u$. Hence, $[t_1t_2]=[ut_\lambda u^{-1}ut_\mu]=[ut_{\lambda+\mu}]$.

Note that $\ell(x)+\ell(y)\leq\ell(z)\leq\ell(x)+\ell(y)+1=\ell([ut_{\lambda+\mu}])=\ell(ut_{\lambda+\mu})$.

If $\ell(z)=\ell(x)+\ell(y)+1=\ell(ut_{\lambda+\mu})$, then the condition $z\preccurlyeq[ut_{\lambda+\mu}]$ implies that $z=ut_{\lambda+\mu}$. This is just case a).

If $\ell(z)=\ell(x)+\ell(y)=\ell(ut_{\lambda+\mu})-1$, then the condition $z\preccurlyeq[ut_{\lambda+\mu}]$ implies that $z\xrightarrow{\sigma_{\gamma+m\delta}}ut_{\lambda+\mu}$ for some $\gamma+m\delta\in R_{\mathrm{re}}^+$. Note that $m\geq 0$ and that $z=\sigma_{\gamma+m\delta}ut_{\lambda+\mu}=$

$\sigma_\gamma u t_{mu^{-1}(\gamma)^\vee+\lambda+\mu} \in W_{\rm af}^-$. Since $\ell(\sigma_{\gamma+m\delta} u t_{\lambda+\mu}) = \ell(z) < \ell(u t_{\lambda+\mu})$, it follows from Lemma 4.3 that $(u t_{\lambda+\mu})^{-1}(\gamma+m\delta) = u^{-1}(\gamma) + \big(m + \langle \lambda+\mu, u^{-1}(\gamma)\rangle\big)\delta \in -R_{\rm re}^+$. Hence, $m + \langle \lambda+\mu, u^{-1}(\gamma)\rangle \le 0$. Since $\lambda+\mu \in \tilde{Q}^\vee$ and $m \ge 0$, we must have $u^{-1}(\gamma) \in R^+$. Therefore,

$$\begin{aligned}
\ell(t_{\lambda+\mu}) - \ell(u) - 1 &= \ell(z)\\
&= \langle m u^{-1}(\gamma)^\vee + \lambda+\mu, -2\rho\rangle - \ell(\sigma_\gamma u)\\
&= \ell(t_{\lambda+\mu}) - m\langle u^{-1}(\gamma)^\vee, 2\rho\rangle - \ell(u\sigma_{u^{-1}\gamma})\\
&\le \ell(t_{\lambda+\mu}) - m\langle u^{-1}(\gamma)^\vee, 2\rho\rangle - \ell(u) + \ell(\sigma_{u^{-1}(\gamma)})\\
&\le \ell(t_{\lambda+\mu}) - m\langle u^{-1}(\gamma)^\vee, 2\rho\rangle - \ell(u) + \langle u^{-1}(\gamma)^\vee, 2\rho\rangle - 1.
\end{aligned}$$

Hence, $(m-1)\langle u^{-1}(\gamma)^\vee, 2\rho\rangle \le 0$. Since $u^{-1}(\gamma) \in R^+$, $\langle u^{-1}(\gamma)^\vee, 2\rho\rangle > 0$. Therefore, $0 \le m \le 1$; that is, $m = 0$ or 1. Denote $\tilde\gamma = u^{-1}(\gamma)$. Note that $\tilde\gamma \in R^+$.

If $m = 0$, then $\tilde\gamma \in R^+, z = u\sigma_{\tilde\gamma} t_{\lambda+\mu}$ and $\ell(u\sigma_{\tilde\gamma}) = \ell(u)+1$. This is just case b).

If $m = 1$, then $\tilde\gamma \in R^+, z = u\sigma_{\tilde\gamma} t_{\lambda+\mu+\tilde\gamma^\vee}$ and $\ell(u\sigma_{\tilde\gamma}) = \ell(u) + 1 - \langle \tilde\gamma^\vee, 2\rho\rangle$. This is just case c).

7.3. **Proof of Proposition 4.20.** Note that we have $z = [t_1 t_2]$ in this case. Since $v_1 t_{\lambda_1} \preccurlyeq \sigma_i t_\lambda \preccurlyeq t_\lambda$, $\lambda \preccurlyeq \lambda_1$ by Corollary 4.9. Hence, $\ell(t_{\lambda_1}) = \ell(t_\lambda) - 2M$ for some $M \ge 0$. Therefore,

$$\begin{aligned}
\ell(x) + \ell(y) = \ell([t_1 t_2]) &= \ell([v_1 t_{\lambda_1} v_1^{-1} v_2 t_{\lambda_2}])\\
&\le \ell(v_1 t_{\lambda_1} v_1^{-1} v_2 t_{\lambda_2})\\
&\le \ell(v_1 t_{\lambda_1}) + \ell(v_1^{-1}) + \ell(v_2 t_{\lambda_2})\\
&= \ell(t_{\lambda_1}) + \ell(v_2 t_{\lambda_2})\\
&\le \ell(t_{\lambda_1}) + \ell(y) = \ell(x) + 1 - 2M + \ell(y).
\end{aligned}$$

Hence, $M = 0$, $\lambda_1 = \lambda$ and $\quad \ell(y) \ge \ell(v_2 t_{\lambda_2}) \ge \ell(x) + \ell(y) - \ell(t_{\lambda_1}) = \ell(y) - 1$.
Hence, there are only the following two possibilities.
Case (i): $\ell(v_2 t_{\lambda_2}) = \ell(y)$, which implies that $v_2 t_{\lambda_2} = y = u t_\mu$.
Case (ii): $\ell(v_2 t_{\lambda_2}) = \ell(y) - 1$.

Due to Lemma 7.2 as below, Case (i) is impossible. It remains to discuss Case (ii). In this case,

$$\begin{aligned}
\ell(x) + \ell(y) = \ell([t_2 t_1]) &= \ell([v_2 t_{\lambda_2} v_2^{-1} v_1 t_\lambda])\\
&\le \ell(v_2 t_{\lambda_2} v_2^{-1} v_1 t_\lambda)\\
&\le \ell(v_2 t_{\lambda_2}) + \ell(v_2^{-1} v_1 t_\lambda)\\
&= \ell(y) - 1 + \ell(t_\lambda) - \ell(v_2^{-1} v_1) = \ell(y) + \ell(x) - \ell(v_2^{-1} v_1).
\end{aligned}$$

Hence, $\ell(v_2^{-1} v_1) = 0$ and therefore $v_1 = v_2$.

Since $\ell(v_2 t_{\lambda_2}) = \ell(y) - 1$ and $v_2 t_{\lambda_2} \preccurlyeq y$, there exists $\gamma + m\delta \in R_{\rm re}^+$ such that $v_2 t_{\lambda_2} = \sigma_{\gamma+m\delta} u t_\mu$. With the same discussion as in the proof of Proposition 4.19, we have $\tilde\gamma = u^{-1}(\gamma) \in R^+$ and $v_2 t_{\lambda_2} = u\sigma_{\tilde\gamma} t_{\mu+m\tilde\gamma^\vee} \in W_{\rm af}^-$. Hence, $v_1 = v_2 = u\sigma_{\tilde\gamma}$ and $z = u\sigma_{\tilde\gamma} t_{\lambda+\mu+m\tilde\gamma^\vee} \in W_{\rm af}^-$. With the same argument as in the proof of Proposition 4.19 again, we can deduce that either $m = 0$ or $m = 1$.

If $m = 0$, then we have $\tilde\gamma \in R^+$ such that case a) holds; that is,

$$v_1 t_{\lambda_1} = u\sigma_{\tilde\gamma} t_\lambda; \quad v_2 t_{\lambda_2} = u\sigma_{\tilde\gamma} t_\mu; \quad z = u\sigma_{\tilde\gamma} t_{\lambda+\mu}; \qquad \ell(u\sigma_{\tilde\gamma}) = \ell(u) + 1.$$

If $m = 1$, then we have $\tilde{\gamma} \in R^+$ such that $v_1 t_{\lambda_1} = u\sigma_{\tilde{\gamma}} t_\lambda$, $v_2 t_{\lambda_2} = u\sigma_{\tilde{\gamma}} t_{\mu+\tilde{\gamma}^\vee}$, $z = u\sigma_{\tilde{\gamma}} t_{\lambda+\mu+\tilde{\gamma}^\vee}$ and $\ell(u\sigma_{\tilde{\gamma}}) = \ell(u) + 1 - \langle \tilde{\gamma}^\vee, 2\rho\rangle$; that is, case b) holds.

Remark 7.1. *The condition "$\langle \mu, \alpha_j\rangle < -\ell(\omega_0)$ for all $j \in I$" does imply that $u\sigma_\gamma t_{\mu+\gamma^\vee}, u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee} \in W_{\rm af}^-$, whenever $\gamma \in \Gamma_2$ and $\lambda \in \tilde{Q}^\vee$.*

Indeed, the statement can be checked directly for the case $|I| = n = 1, 2$. For any $\gamma \in R^+$, write $\gamma^\vee = \sum_i a_i \alpha_i^\vee$. Note that $\ell(\omega_0) = |R^+| \geq 9$ and $a_i \leq 4$ if $n = 3, 4$, and that $\ell(\omega_0) > 12$ and $a_i \leq 6$ if $n \geq 5$ (see e.g. page 66 of [15]*). Hence, $\mu + \gamma^\vee \in \tilde{Q}^\vee$ is regular if $n \geq 3$. In particular, the statement holds.*

Lemma 7.2. Case (i) *in the proof of Proposition 4.20 can never occur.*

Proof. Assume Case (i) holds, then we have $\lambda_1 = \lambda, v_2 t_{\lambda_2} = u t_\mu$ and

$$\begin{aligned}\ell(x) + \ell(y) = \ell([t_2 t_1]) &= \ell([u t_\mu u^{-1} v_1 t_\lambda])\\ &\leq \ell(u t_\mu u^{-1} v_1 t_\lambda)\\ &\leq \ell(u t_\mu) + \ell(u^{-1} v_1 t_\lambda)\\ &= \ell(y) + \ell(t_\lambda) - \ell(u^{-1} v_1) = \ell(y) + \ell(x) + 1 - \ell(u^{-1} v_1).\end{aligned}$$

Therefore, we have either $\ell(u^{-1} v_1) = 0$ or $\ell(u^{-1} v_1) = 1$.

For the former case, we have $v_1 = u$, and therefore $\ell(x) + \ell(y) = \ell([t_1 t_2]) = \ell(u t_{\lambda+\mu}) = \ell(x) + \ell(y) + 1$. This is a contradiction.

For the latter case, $v_1 = u\sigma_j$ for some j. If $\langle \lambda, \alpha_j\rangle \leq \langle \mu, \alpha_j\rangle$, then $\lambda + \mu - \langle \mu, \alpha_j\rangle \alpha_j^\vee \in \tilde{Q}^\vee$. Note that the integer $\langle \mu, \alpha_j\rangle < -\ell(\omega_0)$. Therefore

$$\begin{aligned}\ell(x) + \ell(y) = \ell([t_2 t_1]) &= \ell([u t_\mu u^{-1} u\sigma_j t_\lambda])\\ &\leq \ell(u t_\mu \sigma_j t_\lambda)\\ &= \ell(u\sigma_j t_{\mu - \langle \mu, \alpha_j\rangle \alpha_j^\vee + \lambda})\\ &\leq \ell(u\sigma_j) + \ell(t_{\mu - \langle \mu, \alpha_j\rangle \alpha_j^\vee + \lambda})\\ &= \ell(u\sigma_j) + \langle \mu - \langle \mu, \alpha_j\rangle \alpha_j^\vee + \lambda, -2\rho\rangle\\ &= \ell(u\sigma_j) + \ell(t_{\lambda+\mu}) + 2\langle \mu, \alpha_j\rangle\\ &\leq \ell(\omega_0) + \ell(x) + 1 + \ell(y) + \ell(u) - 2\ell(\omega_0) - 2 < \ell(x) + \ell(y).\end{aligned}$$

If $\langle \lambda, \alpha_j\rangle > \langle \mu, \alpha_j\rangle$, then $\lambda + \mu - \langle \lambda, \alpha_j\rangle \alpha_j^\vee \in \tilde{Q}^\vee$. Therefore,

$$\begin{aligned}\ell(x) + \ell(y) = \ell([t_1 t_2]) &= \ell([u\sigma_j t_\lambda \sigma_j u^{-1} u t_\mu])\\ &\leq \ell(u\sigma_j t_\lambda \sigma_j t_\mu)\\ &= \ell(u t_{\lambda - \langle \lambda, \alpha_j\rangle \alpha_j^\vee + \mu})\\ &\leq \ell(u) + \ell(t_{\lambda - \langle \lambda, \alpha_j\rangle \alpha_j^\vee + \mu})\\ &= \ell(u) + \langle \lambda - \langle \lambda, \alpha_j\rangle \alpha_j^\vee + \mu, -2\rho\rangle\\ &= \ell(u) + \ell(t_{\lambda+\mu}) + 2\langle \lambda, \alpha_j\rangle\\ &\leq \ell(\omega_0) + \ell(x) + 1 + \ell(y) + \ell(u) - 2\ell(\omega_0) - 2 < \ell(x) + \ell(y).\end{aligned}$$

Both cases deduce contradictions. Hence, Case (i) is impossible. □

7.4. **Equivariant cohomology of ΩK.** The affine Kac-Moody group $\mathcal{G}$ possesses a Bruhat decomposition $\bigsqcup_{x \in W_{\rm af}} \mathcal{B} x \mathcal{B}$, where the canonical identification $W_{\rm af} \cong N(\hat{T}_{\mathbb{C}})/\hat{T}_{\mathbb{C}}$ is used. Here $\hat{T}_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(\hat{\mathfrak{h}}_{\mathbb{Z}}^*, \mathbb{C}^*)$ denotes the standard maximal torus

of $\mathcal{G}$, in which $\hat{\mathfrak{h}}_{\mathbb{Z}}$ is the integral form of $\hat{\mathfrak{h}} = \mathfrak{h}_{\text{af}}$ (see e.g. chapter 6 of [24]). The Bruhat decomposition of $\mathcal{G}$ induces a decomposition of $\mathcal{G}/\mathcal{P}_Y$ into Schubert cells: $\mathcal{G}/\mathcal{P}_Y = \bigsqcup_{x\in W_{\text{af}}^Y} \mathcal{B}x\mathcal{P}_Y/\mathcal{P}_Y$. Schubert varieties are the closures of $\mathcal{B}x\mathcal{P}_Y/\mathcal{P}_Y$'s in $\mathcal{G}/\mathcal{P}_Y$. Let $\mathfrak{S}_x^Y$ denote the image of the fundamental class $[\overline{\mathcal{B}x\mathcal{P}_Y/\mathcal{P}_Y}]$ under the canonical map $H_*(\overline{\mathcal{B}x\mathcal{P}_Y/\mathcal{P}_Y}) \to H_*(\mathcal{G}/\mathcal{P}_Y)$. Then $H_*(\mathcal{G}/\mathcal{P}_Y, \mathbb{Z})$ has an additive basis of Schubert homology classes $\{\mathfrak{S}_x^Y \mid x \in W_{\text{af}}^Y\}$. We denote $\mathfrak{S}_x = \mathfrak{S}_x^Y$ wherever there is no confusion. Similarly, the cohomology group $H^*(\mathcal{G}/\mathcal{P}_Y, \mathbb{Z})$ has an additive basis of Schubert cohomology classes $\{\mathfrak{S}^x \mid x \in W_{\text{af}}^Y\}$, where $\langle \mathfrak{S}_x, \mathfrak{S}^y\rangle = \delta_{x,y}$ with respect to the natural pairing.

The standard maximal torus $\hat{T}_{\mathbb{C}}$ of $\mathcal{G}$ has complex dimension $n+2$ with maximal compact sub-torus $\hat{T} = \text{Hom}_{\mathbb{Z}}(\hat{\mathfrak{h}}_{\mathbb{Z}}^*, \mathbb{S}^1)$. With respect to the natural action of $\hat{T}_{\mathbb{C}}$ on $\mathcal{G}/\mathcal{B}$, we consider the equivariant cohomology $H_{\hat{T}}^*(\mathcal{G}/\mathcal{B})$, which is an $\hat{S}$-module with $\hat{S} = S[\hat{\mathfrak{h}}_{\mathbb{Z}}^*] = H_{\hat{T}}^*(\text{pt})$. Note that the 1-dimensional sub-torus $\mathbb{C}^*$, which comes from the degree derivation d, acts on $\mathcal{G}/\mathcal{B}$ trivially. As a consequence, $\hat{S} = \mathbb{Q}[d, \delta, \alpha_1, \cdots, \alpha_n]$ while the equivariant Schubert structure constants are polynomials in $\mathbb{Q}[\delta, \alpha_1, \cdots, \alpha_n]$ only. Since we are concerned with the non-trivial part of the $\hat{T}_{\mathbb{C}}$-action only, we denote $\hat{S} = \mathbb{Q}[\delta, \alpha_1, \cdots, \alpha_n] = \mathbb{Q}[\alpha_0, \alpha_1, \cdots, \alpha_n]$ by abusing notations. $H_{\hat{T}}^*(\mathcal{G}/\mathcal{B})$ is an $\hat{S}$-module spanned by the basis of equivariant Schubert classes, which we also denote as $\{\mathfrak{S}^x \mid x \in W_{\text{af}}\}$ simply. Via the embedding $\pi^* : H_{\hat{T}}^*(\mathcal{G}/\mathcal{P}_Y) \hookrightarrow H_{\hat{T}}^*(\mathcal{G}/\mathcal{B})$ induced by the natural projection $\pi : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_Y$, the equivariant cohomology $H_{\hat{T}}^*(\mathcal{G}/\mathcal{P}_Y)$ is also an $\hat{S}$-module spanned by the basis of equivariant Schubert classes $\{\mathfrak{S}^x \mid x \in W_{\text{af}}^Y\}$. As a consequence, equivariant Schubert structure constants for $\mathcal{G}/\mathcal{P}_Y$ are covered by equivariant Schubert structure constants for $\mathcal{G}/\mathcal{B}$.

Remark 7.3. *For $\mathfrak{S}^x, \mathfrak{S}^y \in H_{\hat{T}}^*(\mathcal{G}/\mathcal{B})$, $\mathfrak{S}^x\mathfrak{S}^y = \sum_{z\in W_{\text{af}}} p_{x,y}^z \mathfrak{S}^z$. The equivariant Schubert structure constant $p_{x,y}^z$ is a polynomial in $\hat{S}$. In terms of combination of rational functions, one has $p_{x,y}^z = \sum_{v\in W_{\text{af}}} d_{x,v}d_{y,v}c_{z,v}$ (see e.g. chapter 11 of* [24]*).*

Let $L_{\text{an}}K = \{f \in \mathcal{G} \mid f(\mathbb{S}^1) \subset K\}$ and $\Omega_{\text{an}}K = \{f \in L_{\text{an}}K \mid f(1_{\mathbb{S}^1}) = 1_K\}$. Note that each $f \in \mathcal{G}$ can be written as $f(t) = f_K(t)\cdot f_P(t)$ for some unique $f_K \in \Omega_{\text{an}}K$ and $f_P \in \mathcal{P}_0$. Therefore we can realize $\mathcal{G}/\mathcal{P}_0$ as $\Omega_{\text{an}}K$, which is homotopy-equivalent to ΩK, via the ($L_{\text{an}}K$-equivariant) homeomorphism $\mathcal{G}/\mathcal{P}_0 \to \Omega_{\text{an}}K$ (see [36] and references therein for more details). Since we are concerned with properties at the level of (co)homology only, we do not distinguish between $\Omega_{\text{an}}K$ and ΩK. The Bruhat decomposition of $\mathcal{G}/\mathcal{P}_0$ readily gives a Bruhat decomposition of ΩK. As a consequence and by abusing notations, we know that $H_*(\Omega K, \mathbb{Z})$ (resp. $H^*(\Omega K, \mathbb{Z})$) has an additive $\mathbb{Z}$-basis of Schubert (co)homology classes $\{\mathfrak{S}_x(\text{resp. } \mathfrak{S}^x) \mid x \in W_{\text{af}}^-\}$.

The (non-trivial part of the) $\hat{T}$-action on $\mathcal{G}/\mathcal{P}_0$ corresponds to the natural $\mathbb{S}^1\times T$ action on ΩK, which consists of the rotation action of $\mathbb{S}^1$ on ΩK and the action of T on ΩK by pointwise conjugation. By considering the T-action only, we obtain the evaluation maps $\text{ev} : H_{\hat{T}}^*(\mathcal{G}/\mathcal{P}_0) \to H_T^*(\mathcal{G}/\mathcal{P}_0)$ and $ev : \hat{S} = H_{\hat{T}}^*(\text{pt}) \to H_T^*(\text{pt}) = S$, where the T-equivariant cohomology $H_T^*(\mathcal{G}/\mathcal{P}_0)$ is an S-module with $S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$. The image of the null root $\delta = \alpha_0 + \theta$ in S is 0. More precisely, we have $H_{\hat{T}}^*(\mathcal{G}/\mathcal{P}_0) = \text{Span}_{\hat{S}}\{\hat{\mathfrak{S}}^x \mid x \in W_{\text{af}}^-\}$ and $H_T^*(\mathcal{G}/\mathcal{P}_0) = \text{Span}_S\{\mathfrak{S}^x \mid x \in W_{\text{af}}^-\}$. Let $f = f(\alpha_0, \alpha_1, \cdots, \alpha_n) \in \hat{S}$, then we have $ev(f) = f(-\theta, \alpha_1, \cdots, \alpha_n) \in S$ and $\text{ev}(f\hat{\mathfrak{S}}^x) = ev(f)\mathfrak{S}^x$.

Remark 7.4. $\mathfrak{S}^x\mathfrak{S}^y = \sum_{z\in W_{\mathrm{af}}^-}\tilde{p}_{x,y}^z\mathfrak{S}^z$. *The T-equivariant Schubert structure constant $\tilde{p}_{x,y}^z$ is a polynomial in S. It follows from Remark 7.3 and Lemma 4.22 that $\tilde{p}_{x,y}^z = \sum_{v\in W_{\mathrm{af}}^-} d_{x,[v]}d_{y,[v]}c_{z,[v]}$ as combination of rational functions.*

Remark 7.5. *The T-equivariant Schubert structure constant $p_{u,v}^w$ for G/B can also be expressed in terms of $c_{u,v}$ and $d_{u,v}$. The polynomial $p_{u,v}^w$ is given by $p_{u,v}^w = \sum_{v_1\in W} d_{u,v_1}d_{v,v_1}c_{w,v_1}$ as combination of rational functions (see e.g.* [24]*).*

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